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Guaranteed Cost Robust Output Feedback Control Design for Fractional-Order Uncertain Neutral Delay Systems

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Abstract

This paper investigates the stabilization of uncertain neutral-type delay fractional systems. Both static and dynamic output feedback control design methods are proposed. A guaranteed cost-based feedback control design is considered and closed-loop robust stability is investigated using the Lyapunov theorem. Asymptotic stability conditions of the closed-loop system is described in terms of linear matrix inequalities (LMIs). A simulation study, encompassing a numerical example of a fractional order system and a two stage chemical reactor, is considered to assess the performance and validity of the proposed control design.

Keywords: Stability, Fractional calculus, Guaranteed cost control, Neutral delay, Feedback.

1. Introduction

Delays in dynamic systems are ubiquitous and might degrade the performance of the closed-loop system considerably. Therefore, dealing with the effects of delay has been a popular research topic [1–3]. This paper studies the case of neutral systems, that is, systems that include delays in the states and their derivatives [4]. A number of physical systems can be modeled by neutral delay-differential equations [5] and, hence, studying their stability and stabilization is pertinent [6–8].

Ignoring the uncertainties in the dynamical models, such as parameter perturbations, can significantly compromise the controller performance. Thus, explicitly dealing with the effects of uncertainties is key for guaranteeing system stability and ensuring good dynamic performance in the presence of those disturbances. Designing robust controllers for uncertain systems has been well discussed in the literature [9–12]. When systems with uncertainties are considered, a robust controller that simultaneously ensures adequate system performance and stability must be designed. Guaranteed cost control (GCC) is a valid approach to deal with such problems [13], where GCC provides an upper bound for a prescribed index in the presence of uncertainty. The linear matrix inequality (LMI) is an effective technique to deal with the GCC of dynamic systems [14–18]. For instance, authors in [14] studied the GCC of uncertain neutral delay systems. Later, this approach was utilized for uncertain tele-robotic delayed systems' stabilization [15]. In another study [16], the GCC was used to investigate complex network's synchronization. Authors in [17] addressed the GCC of nonlinear systems with uncertainties and mixed delays, while in another study [18], the GCC was discussed for continuous-time nonlinear delayed systems. The GCC problem of delayed cellular neural

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networks (NNs) was addressed in [19] using the Razumikhin theorem. Later in [20], the previous works' method was enhanced using the Lyapunov-Krasovskii functional approach. Authors in [21] utilized the Lyapunov-Krasovksii approach to investigate the GCC of delayed cellular NNs, while in [22] the GCC of delayed singular systems was addressed. In another study [23], a feedback control scheme based on GCC was considered for uncertain neutral systems. Nonetheless, despite the considerable achievements of the foregoing research efforts, such works were only limited to integer-order systems.

Fractional-order (FO) models have been adopted in diverse research fields. [24]. A key characteristic of FO descriptions in comparison with the integer-order systems is their infinite memory, resulting in improving the accuracy of modeling. some efficient applications of the FO models are electro-mechanical systems [25–27], physics [28–30], image processing [31], and biology [32, 33]. However, FO models demand for more intricate mathematical treatment, resulting in more a complicated stabilization study than their integer-order counterparts [34–38]. Authors in [39] investigated the GCC of FO delayed NNs systems, while later in [40, 41] GCC schemes based on state feedback were developed for linear time-delayed FO systems. GCC schemes for time-delayed linear FO systems were developed in the form of state-feedback. Control schemes based on state feedback were developed for linear time-delayed FO systems. However, in the case of FO uncertain neutral type-delay systems, the GCC problem has not yet been sufficiently studied, which leads to the first issue worth addressing in this paper.

The Lyapunov concept has broadly been adopted to study the stability and stabilization problems of the dynamical systems [42]. The majority of studies on the stabilization of the time-delayed system [43–46] have considered a state feedback controller, particularly for the neutral-type delay systems [47–50]. However, in practice all of the states in a system are not always available for measurement. Hence, to mitigate this problem, this paper will consider an output-feedback control (OFC) design [51–54], which forms another motivation for us to study the problem. The OFC is generally addressed in different manners, namely the static and the dynamic. The dynamic OFC is more effective than the other for improving the closed-loop transient response. However, the design of dynamic output-feedback controller is more complex. According to our knowledge, the GCC of uncertain FO neutral systems using the dynamic output feedback technique has not been yet explored. Tables 1 and 2 summarize the key differences with respect of FO model characteristics and the designed controller. Therefore, the static and dynamic output-feedback controller design methods for such systems are proposed in this paper. Our approach relies on the Lyapunov theorem, which provides LMI-based closed-loop robust stability conditions that can be solved efficiently.

Table 1: Comparison in terms of different aspects of FO model

Related works	FO model properties		Delay type		
	Nonlinearity	Parametric uncertainty	Constant	Varying	Neutral
[38]		*			
[41]		*	*		
[39]	*	*	*		
[43]	*	*			
[46]				*	
[53]		*			
[54]		*			
Current work		*	*		*

The remainder of the article is organized as follows. Some preliminary concepts are provided in Section 2. The

Table 2: Comparison in terms of the controller types

Related works	Controller type			
	State-feedback	Output-feedback		Optimality
		Static	Dynamic	
[38]		*		
[41]	*			*
[39]	*			*
[43]	*			
[46]	*			
[53]			*	
[54]			*	
Current work		*	*	*

theoretical findings are then presented in Section 3. Section 4 is devoted to simulation results that demonstrate the effectiveness of the main results. Finally, Section 5 concludes the paper.

Throughout the paper, a diagonal matrix is shown as $diag\{\cdot\}$. The $n \times n$ identity matrix is I, $*$ and T stand for the symmetric part of a square matrix and transpose of a vector/matrix, respectively, $\mathbb{R}^n$ signifies the n-dimensional linear vector space over the real numbers. Euclidean norm is denoted by $\|.\|$, and $\mathbb{R}^+$ stands for the subset of real numbers that are strictly greater than zero.

2. Fundamental Concepts

Here, first, we shall present some definitions and lemmas from the literature to develop the main results of this paper. Then, the control problem will be stated.

Definition 1 [55]. The integral of order q for any continuous function $\upsilon(t)$ is defined as

$$I_{t_0,t}^{q}\upsilon(t) = \frac{1}{\Gamma\ (q)} \int\limits_{t_0}^{t} (t-\theta)^{q-1}\upsilon(\theta)\mathrm{d}\theta,\ q > 0, \tag{1}$$

where $\Gamma(q)$ is the Gamma function.

Definition 2 [55]. For any continuous function $\upsilon(t)$, the Caputo derivative of fractional-order $q \in \mathbb{R}$ is expressed as

$${}^{C}\mathfrak{D}^{q}\upsilon(t) = I_{t_0,t}^{k-q}\frac{d^k}{dt^k}\upsilon(t) = \frac{1}{\Gamma(k-q)} \int\limits_{t_0}^{t} (t-\theta)^{k-q-1}\upsilon^{(k)}(\theta)\mathrm{d}\theta, \tag{2}$$

where $k \in \mathbb{Z}^+$ and $0 \leq k-1 \leq q < k$.

Lemma 1 [56]: For the matrices Λ_1, Λ_2 and Λ_3, where $\Lambda_1 = {\Lambda_1}^T$ and $\Lambda_2 > 0$, then $\Lambda_1 + {\Lambda_3}^T{\Lambda_2}^{-1}\Lambda_3 < 0$ if and only if

$$\begin{pmatrix} \Lambda_1 & {\Lambda_3}^T \\ \Lambda_3 & -\Lambda_2 \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -\Lambda_2 & \Lambda_3 \\ {\Lambda_3}^T & \Lambda_1 \end{pmatrix} < 0. \tag{3}$$

Lemma 2 [57]: For real matrices $\mathscr{G}, \mathscr{E}, \mathcal{H}$ with appropriate dimensions, and any matrix $\mathfrak{L}$ satisfying $\mathfrak{L}^T(t)\mathfrak{L}(t) \leq I$, if and only if there exists a constant $q \in \mathbb{R}^+$ such that

$$\mathscr{G} + q^{-1}\mathscr{E}^T\mathscr{E} + q\mathcal{H}\mathcal{H}^T < 0. \tag{4}$$

then

$$\mathscr{G} + \mathcal{H}\mathfrak{L}(t)\mathscr{E} + \mathscr{E}^T\mathfrak{L}^T(t)\mathcal{H}^T < 0, \tag{5}$$

is valid.

Lemma 3 [58]: Consider a differentiable vector-valued function $\mathcal{N}(t) \in \mathbb{R}^n$ and a symmetric positive definite matrix $\mathscr{G} \in \mathbb{R}^{n\times n}$. Then, for $q \in (0,1)$, we have

$$ {}^{C}\mathfrak{D}^q\left(\mathcal{N}^T(t)\mathscr{G}\mathcal{N}(t)\right) \leq \left(\mathcal{N}^T(t)\mathscr{G}\right){}^{C}\mathfrak{D}^q\mathcal{N}(t) + \left({}^{C}\mathfrak{D}^q\mathcal{N}(t)\right)^T\mathscr{G}\mathcal{N}(t), \quad t \geq t_0, \tag{6} $$

Lemma 4 [59] (Razumikhin stability theorem): For the given system

$$ {}^{C}\mathfrak{D}^q\mathscr{X}(t) = g(t, \mathscr{X}_t), \tag{7} $$

where $\mathscr{X}_t = \mathscr{X}(t+\theta)$, $-\mathfrak{h} \leq \theta \leq 0$. Suppose the nondecreasing continuous functions $\{\xi_1, \xi_2, \xi_3\} : \mathbb{R}^+ \to \mathbb{R}^+$, and also the functions $\xi_1(z)$ and $\xi_2(z)$ being positive for $z > 0$, with $\xi_1(0) = \xi_2(0) = 0$ and ξ_2 is strictly increasing. If there exists a constant $\alpha > 1$ and a continuously differentiable function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ such that

(1) $\xi_1(\|\mathscr{X}\|) \leq V(t, \mathscr{X}) \leq \xi_2(\|\mathscr{X}\|)$,

(2) ${}^{C}\mathfrak{D}^q V(t, \mathscr{X}(t)) \leq -\xi_3(\|\mathscr{X}(t)\|)\ if \ \ V(t+\delta, \mathscr{X}(t+\delta)) \leq \alpha V(t, \mathscr{X}(t)),\ \ t \geq 0, \forall\delta \in [-\varrho, 0]$,

the zero solution of system (7) is asymptotically stable.

Lemma 5 [55]: If $\upsilon(t) \in \mathbb{C}^n([0, +\infty), \mathbb{R})$, and $n-1 < q < n$, where $n \geq 1, n \in \mathbb{Z}^+$, then

$$ I_t^q({}^{C}\mathfrak{D}_t^q\,\upsilon(t)) = \upsilon(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}\upsilon^{(k)}(0). \tag{8} $$

In particular, when $0 < q < 1$, the upper form turns into

$$ I_t^q({}^{C}\mathfrak{D}_t^q\,\upsilon(t)) = \upsilon(t) - \upsilon(0). \tag{9} $$

This paper considers FO uncertain neutral systems in the following state-space representation with $q \in (0,1)$

$$ \begin{aligned} &{}^{C}\mathfrak{D}^q\upsilon(t) = (\mathcal{A}+\Delta\mathcal{A})\upsilon(t) + (\mathcal{A}_d+\Delta\mathcal{A}_d)\upsilon(t-\varrho) + (\mathcal{A}_0+\Delta\mathcal{A}_0){}^{C}\mathfrak{D}^q\upsilon(t-\varrho) + (\mathcal{B}+\Delta\mathcal{B})w(t), \\ &\upsilon(t) = f(t), \quad t \in [-\varrho, 0], \\ &\mathcal{Y}(t) = \mathcal{G}\upsilon(t), \end{aligned} \tag{10} $$

where $\mathcal{Y} \in \mathbb{R}^{\,v}$ and $\upsilon \in \mathbb{R}^n$ respectively denote the output and state vectors, $w \in \mathbb{R}^m$ represents the input vector, ϱ shows the constant delay, and $\mathcal{A}, \mathcal{A}_d, \mathcal{A}_0, \mathcal{B}, \mathcal{G}$ are known real matrices with compatible dimensions. Additionally, the uncertainty terms are expressed by

$$ [\Delta\mathcal{A} \quad \Delta\mathcal{A}_d \quad \Delta\mathcal{A}_0 \quad \Delta\mathcal{B}] = \mathcal{H}\aleph(t)[\mathcal{E}_0 \quad \mathcal{E}_1 \quad \mathcal{E}_2 \quad \mathcal{E}_3], \tag{11} $$

in which the known real matrices $\mathcal{H}, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ are of compatible dimensions and the time-varying parametric uncertainties are reflected onto $\aleph(t)$, which satisfies $\aleph^T(t)\aleph(t) \leq I$.

Next, for system (10), the static and dynamic output-feedback control laws will be determined to guarantee the closed-loop asymptotical stability with the following cost function to be minimized.

$$ \mathcal{M} = \frac{1}{\Gamma(q)}\int_0^{\mathfrak{n}}(\mathfrak{n}-\sigma)^{q-1}\Big(\upsilon^T(\sigma)Q_1\upsilon(\sigma) + w^T(\sigma)Q_2 w(\sigma)\Big)\mathrm{d}\sigma,\ \ \forall\,\mathfrak{n} > 0, \tag{12} $$

where Q_1 and Q_2 are positive- definite matrices.

Remark 1: Note that if we set $q = 1$, then (12) becomes the standard definition of linear quadratic cost function in integer-order form. Since we employ Lemma 3, the results of this paper is only valid when $q \in (0, 1)$.

Definition 3 [14]. Consider the FO system (10) with the cost function (12). Given the existence of a control law $w^\star(t)$ and a positive scalar $\mathcal{M}^\star$ for all permissible uncertainties and specified delay, the overall asymptotical stability is guaranteed. Besides, if $\mathcal{M} < \mathcal{M}^\star$ holds, then $\mathcal{M}^*$ and $w^\star(t)$ are the guaranteed cost value and control law for the system (10), respectively.

3. Theoretical Results

In this section we shall propose the delay independent static and dynamic output-feedback GCC design methods for system (10).

3.1. Static output-feedback control solution

Consider system (10) with the controller of the following form

$$w(t) = L\mathcal{Y}(t). \tag{13}$$

The closed-loop system dynamics can be expressed as

$$\begin{aligned} &{}^{C}\mathfrak{D}^{q}\upsilon(t) = (\mathcal{A} + \mathcal{B}L\mathcal{G} + \Delta\mathcal{A} + \Delta\mathcal{B}L\mathcal{G})\upsilon(t) + (\mathcal{A}_d + \Delta\mathcal{A}_d)\upsilon(t-\varrho) + (\mathcal{A}_0 + \Delta\mathcal{A}_0)^{C}\mathfrak{D}^{q}\upsilon(t-\varrho), \\ &\upsilon(t) = f(t), \quad t \in [-\varrho, 0]. \end{aligned} \tag{14}$$

Theorem 1 provides satisfactory conditions for the asymptotic stability of (10), in the presence of control law (13).

Theorem 1: Consider system (10) with the matrices Q_1 and Q_2 as given in (12). Given the existence of a constant $\gamma > 0$, appropriately dimensioned matrix $\mathcal{X}$, and a symmetric positive definite matrix $\tilde{\mathcal{F}}$ satisfying

$$\phi = \begin{pmatrix} \phi_{11} & \mathcal{A}_d\tilde{\mathcal{F}} & \phi_{13} & \phi_{14} & \phi_{15} & \gamma\mathcal{H} & \tilde{\mathcal{F}}Q_1 & \mathcal{X}^TQ_2 \\ * & -\tilde{\mathcal{F}} & -\tilde{\mathcal{F}}{\mathcal{A}_d}^T & \tilde{\mathcal{F}}{\mathcal{A}_d}^T & \tilde{\mathcal{F}}{\mathcal{E}_1}^T & 0 & 0 & 0 \\ * & * & \phi_{33} & \tilde{\mathcal{F}} + \tilde{\mathcal{F}}{\mathcal{A}_0}^T & \tilde{\mathcal{F}}{\mathcal{E}_2}^T & -\gamma\mathcal{H} & 0 & 0 \\ * & * & * & -2\tilde{\mathcal{F}} & 0 & \gamma\mathcal{H} & 0 & 0 \\ * & * & * & * & -\gamma I & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & * & * & -Q_1 & 0 \\ * & * & * & * & * & * & * & -Q_2 \end{pmatrix} < 0, \tag{15}$$

where $\phi_{11} \triangleq \mathcal{A}\tilde{\mathcal{F}} + \tilde{\mathcal{F}}\mathcal{A}^T + \mathcal{B}\mathcal{X} + \mathcal{X}^T\mathcal{B}^T + \tilde{\mathcal{F}}$, $\phi_{13} \triangleq \mathcal{A}_0\tilde{\mathcal{F}} - \tilde{\mathcal{F}}\mathcal{A}^T - \mathcal{X}^T\mathcal{B}^T$, $\phi_{14} \triangleq \tilde{\mathcal{F}}\mathcal{A}^T + \mathcal{X}^T\mathcal{B}^T$, $\phi_{15} \triangleq \tilde{\mathcal{F}}{\mathcal{E}_0}^T + \mathcal{X}^T{\mathcal{E}_3}^T$, and $\phi_{33} \triangleq -\tilde{\mathcal{F}}{\mathcal{A}_0}^T - \mathcal{A}_0\tilde{\mathcal{F}}$, then the system (10) is asymptotically stable via the following output feedback gain

$$L = \mathcal{X}\tilde{\mathcal{F}}^{-1}\mathcal{G}^T(\mathcal{G}\,\mathcal{G}^T)^{-1}, \tag{16}$$

and the guaranteed cost value can be calculated as

$$\mathcal{M}^\star = \lambda_{\max}(\tilde{\mathcal{F}}^{-1})(\|f\|)^2. \tag{17}$$

Proof: Consider the Lyapunov function

$$V(\upsilon(t)) = \upsilon^T(t)\mathcal{F}\upsilon(t). \tag{18}$$

Considering Lemma 3, the FO derivation of (18) along the system trajectory (10) satisfies

$$\begin{aligned}
&{}^C\mathfrak{D}^q V(\upsilon(t)) + \upsilon^T(t)Q_1\upsilon(t) + \upsilon^T(t)(L\mathcal{G})^T Q_2(L\mathcal{G})\upsilon(t) \\
&\leq \upsilon^T(t)\mathcal{F}\,{}^C\mathfrak{D}^q\upsilon(t) \ + \left({}^C\mathfrak{D}^q\upsilon(t)\right)^T \mathcal{F}\upsilon(t) \\
&\quad + \upsilon^T(t)Q_1\upsilon(t) + \upsilon^T(t)(L\mathcal{G})^T Q_2(L\mathcal{G})\upsilon(t) \\
&\leq \upsilon^T(t)\left(\mathcal{F}\mathcal{A} + \mathcal{F}\mathcal{B}L\mathcal{G} + (L\mathcal{G})^T\mathcal{B}^T\mathcal{F} + \mathcal{A}^T\mathcal{F}\right)\upsilon(t) \\
&\quad + \upsilon^T(t)\left(\mathcal{F}\Delta\mathcal{B}L\mathcal{G} + \mathcal{F}\Delta\mathcal{A} + (L\mathcal{G})^T\Delta\mathcal{B}^T\mathcal{F} + \Delta\mathcal{A}^T\mathcal{F}\right)\upsilon(t) \\
&\quad + \upsilon^T(t)\left(\mathcal{F}\mathcal{A}_d + \mathcal{F}\Delta\mathcal{A}_d\right)\upsilon(t-\varrho) \\
&\quad + \upsilon^T(t-\varrho)\left({\mathcal{A}_d}^T\mathcal{F} + \Delta{\mathcal{A}_d}^T\mathcal{F}\right)\upsilon(t) \\
&\quad + \upsilon^T(t)\left(\mathcal{F}\mathcal{A}_0 + \mathcal{F}\Delta\mathcal{A}_0\right){}^C\mathfrak{D}^q\upsilon(t-\varrho) \\
&\quad +{}^C\mathfrak{D}^q\upsilon^T(t-\varrho)\left({\mathcal{A}_0}^T\mathcal{F} + \Delta{\mathcal{A}_0}^T\mathcal{F}\right)\upsilon(t) \\
&\quad + \upsilon^T(t)(L\mathcal{G})^T Q_2(L\mathcal{G})\,\upsilon(t) + \upsilon^T(t)Q_1\upsilon(t).
\end{aligned} \tag{19}$$

Considering (18), and using Lemma 4, whenever $\upsilon(t)$ satisfies

$$\alpha V(t, \upsilon(t)) > V(t+\delta, \upsilon(t+\delta)), \quad -\varrho \leq \delta \leq 0, \tag{20}$$

for some $\alpha > 1$, one gets

$$\alpha\upsilon^T(t)\mathcal{F}\upsilon(t) - \upsilon^T(t-\varrho)\mathcal{F}\upsilon(t-\varrho) \geq 0. \tag{21}$$

Substituting (21) in (19), and adding the following null equation to the right hand side of (19)

$$\begin{aligned}
2\left(-{}^C\mathfrak{D}^q\upsilon^T(t-\varrho) + {}^C\mathfrak{D}^q\upsilon^T(t)\right)\mathcal{F}\left(-{}^C\mathfrak{D}^q\upsilon(t) + (\mathcal{A} + \mathcal{B}L\mathcal{G} + \Delta\mathcal{A} + \Delta\mathcal{B}L\mathcal{G})\upsilon(t)\right. \\
\left. +(\mathcal{A}_d + \Delta\mathcal{A}_d)\upsilon(t-\varrho) + (\mathcal{A}_0 + \Delta\mathcal{A}_0)^C\mathfrak{D}^q\upsilon(t-\varrho)\right) = 0.
\end{aligned}$$

yields

$$
\begin{aligned}
&{}^{C}\mathfrak{D}^{q}V(\upsilon(t))+\upsilon^{T}(t)Q_{1}\upsilon(t)+\upsilon^{T}(t)(L\mathcal{G})^{T}Q_{2}(L\mathcal{G})\upsilon(t)\\
&\leq \upsilon^{T}(t)\left(\mathcal{F}\mathcal{A}+\mathcal{F}\mathcal{B}L\mathcal{G}+(L\mathcal{G})^{T}\mathcal{B}^{T}\mathcal{F}+\alpha\mathcal{F}+\mathcal{A}^{T}\mathcal{F}\right)\upsilon(t)\\
&\quad+\upsilon^{T}(t)\left(\mathcal{F}\Delta\mathcal{A}+\Delta\mathcal{A}^{T}\mathcal{F}+\mathcal{F}\Delta\mathcal{B}L\mathcal{G}+(L\mathcal{G})^{T}\Delta\mathcal{B}^{T}\mathcal{F}\right)\upsilon(t)\\
&\quad+\upsilon^{T}(t)\left(\mathcal{F}\mathcal{A}_{d}+\mathcal{F}\Delta\mathcal{A}_{d}\right)\upsilon(t-\varrho)+\upsilon^{T}(t-\varrho)\left(\mathcal{A}_{d}{}^{T}\mathcal{F}+\Delta\mathcal{A}_{d}{}^{T}\mathcal{F}\right)\upsilon(t)\\
&\quad+\upsilon^{T}(t)\left(\mathcal{F}\mathcal{A}_{0}+\mathcal{F}\Delta\mathcal{A}_{0}\right){}^{C}\mathfrak{D}^{q}\upsilon(t-\varrho)+{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t-\varrho)\left(\mathcal{A}_{0}{}^{T}\mathcal{F}+\Delta\mathcal{A}_{0}{}^{T}\mathcal{F}\right)\upsilon(t)\\
&\quad+\upsilon^{T}(t)Q_{1}\upsilon(t)+\upsilon^{T}(t)(L\mathcal{G})^{T}Q_{2}(L\mathcal{G})\,\upsilon(t)-\upsilon^{T}(t-\varrho)\mathcal{F}\upsilon(t-\varrho)\\
&\quad+2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t)(\mathcal{F}\mathcal{A}+\mathcal{F}\mathcal{B}L\mathcal{G})\upsilon(t)-2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t)\mathcal{F}\,{}^{C}\mathfrak{D}^{q}\upsilon(t)\\
&\quad+2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t)(\mathcal{F}\Delta\mathcal{A}+\mathcal{F}\Delta\mathcal{B}L\mathcal{G})\upsilon(t)+2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t)(\mathcal{F}\mathcal{A}_{d}+\mathcal{F}\Delta\mathcal{A}_{d})\upsilon(t-\varrho)\\
&\quad+2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t)(\mathcal{F}\mathcal{A}_{0}+\mathcal{F}\Delta\mathcal{A}_{0})\,\mathfrak{D}^{q}\upsilon(t-\varrho)+2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t-\varrho)\mathcal{F}\,{}^{C}\mathfrak{D}^{q}\upsilon(t)\\
&\quad-2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t-\varrho)(\mathcal{F}\mathcal{A}+\mathcal{F}\mathcal{B}L\mathcal{G})\upsilon(t)-2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t-\varrho)(\mathcal{F}\Delta\mathcal{A}+\mathcal{F}\Delta\mathcal{B}L\mathcal{G})\upsilon(t)\\
&\quad-2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t-\varrho)(\mathcal{F}\mathcal{A}_{d}+\mathcal{F}\Delta\mathcal{A}_{d})\upsilon(t-\varrho)-2\,{}^{C}\mathfrak{D}^{q}\upsilon^{T}(t-\varrho)(\mathcal{F}\mathcal{A}_{0}+\mathcal{F}\Delta\mathcal{A}_{0})\,{}^{C}\mathfrak{D}^{q}\upsilon(t-\varrho)\\
&\leq \eta^{T}(t)\,\psi\eta(t),
\end{aligned}
$$

where $\eta^{T}(t) \triangleq \left[\upsilon^{T}(t),\quad \upsilon^{T}(t-\varrho),\quad {}^{C}\mathfrak{D}^{q}\upsilon^{T}(t-\varrho),\quad {}^{C}\mathfrak{D}^{q}\upsilon^{T}(t)\right]$ and $\psi \triangleq \begin{pmatrix}\psi_{11} & \psi_{12} & \psi_{13} & \psi_{14}\\ * & -\mathcal{F} & -\mathcal{A}_{d}{}^{T}\mathcal{F} & \mathcal{A}_{d}{}^{T}\mathcal{F}\\ * & * & \psi_{33} & \psi_{34}\\ * & * & * & -2\mathcal{F}\end{pmatrix}$ with $\psi_{11} \triangleq \mathcal{F}\mathcal{A}+\mathcal{A}^{T}\mathcal{F}+\mathcal{F}\mathcal{B}L\mathcal{G}+(L\mathcal{G})^{T}\mathcal{B}^{T}\mathcal{F}+\alpha\mathcal{F}+Q_{1}+(L\mathcal{G})^{T}Q_{2}(L\mathcal{G})+\mathcal{F}\Delta\mathcal{A}+\Delta\mathcal{A}^{T}\mathcal{F}+\mathcal{F}\Delta\mathcal{B}L\mathcal{G}+(L\mathcal{G})^{T}\Delta\mathcal{B}^{T}\mathcal{F}$, $\psi_{12} \triangleq \mathcal{F}\mathcal{A}_{d}+\mathcal{F}\Delta\mathcal{A}_{d}$, $\psi_{13} \triangleq \mathcal{F}\mathcal{A}_{0}+\mathcal{F}\Delta\mathcal{A}_{0}-(L\mathcal{G})^{T}\mathcal{B}^{T}\mathcal{F}-(L\mathcal{G})^{T}\Delta\mathcal{B}^{T}\mathcal{F}-\mathcal{A}^{T}\mathcal{F}-\Delta\mathcal{A}^{T}\mathcal{F}$, $\psi_{14} \triangleq \mathcal{A}^{T}\mathcal{F}+(L\mathcal{G})^{T}\mathcal{B}^{T}\mathcal{F}+\Delta\mathcal{A}^{T}\mathcal{F}+(L\mathcal{G})^{T}\Delta\mathcal{B}^{T}\mathcal{F}$, $\psi_{33} \triangleq -\mathcal{F}\mathcal{A}_{0}-\mathcal{A}_{0}{}^{T}\mathcal{F}$, and $\psi_{34} \triangleq \mathcal{F}+\mathcal{A}_{0}{}^{T}\mathcal{F}$.

We conclude that $\psi \leq 0$ can be reorganized in the form of

$$
\begin{aligned}
\Theta = &\begin{pmatrix}\Theta_{11} & \mathcal{F}\mathcal{A}_{d} & \mathcal{F}\mathcal{A}_{0}-(L\mathcal{G})^{T}\mathcal{B}^{T}\mathcal{F}-\mathcal{A}^{T}\mathcal{F} & \mathcal{A}^{T}\mathcal{F}+(L\mathcal{G})^{T}\mathcal{B}^{T}\mathcal{F}\\ * & -\mathcal{F} & -\mathcal{A}_{d}{}^{T}\mathcal{F} & \mathcal{A}_{d}{}^{T}\mathcal{F}\\ * & * & -\mathcal{F}\mathcal{A}_{0}-\mathcal{A}_{0}{}^{T}\mathcal{F} & \mathcal{F}+\mathcal{A}_{0}{}^{T}\mathcal{F}\\ * & * & * & -2\mathcal{F}\end{pmatrix}\\
&+\begin{pmatrix}\mathcal{F}\mathcal{H}\\ 0\\ -\mathcal{F}\mathcal{H}\\ \mathcal{F}\mathcal{H}\end{pmatrix}\aleph(t)\begin{pmatrix}\mathcal{E}_{0}+\mathcal{E}_{3}L\mathcal{G} & \mathcal{E}_{1} & \mathcal{E}_{2} & 0\end{pmatrix}\\
&+\begin{pmatrix}\mathcal{E}_{0}{}^{T}+\mathcal{G}^{T}L^{T}\mathcal{E}_{3}{}^{T}\\ \mathcal{E}_{1}{}^{T}\\ \mathcal{E}_{2}{}^{T}\\ 0\end{pmatrix}\aleph^{T}(t)\begin{pmatrix}\mathcal{H}^{T}\mathcal{F} & 0 & -\mathcal{H}^{T}\mathcal{F} & \mathcal{H}^{T}\mathcal{F}\end{pmatrix}\\
&<0,
\end{aligned}
$$

where $\Theta_{11} \triangleq \mathcal{F}\mathcal{A} + (L\mathcal{G})^T \mathcal{B}^T \mathcal{F} + \mathcal{F}\mathcal{B}L\mathcal{G} + \mathcal{A}^T \mathcal{F} + \alpha \mathcal{F} + Q_1 + (L\mathcal{G})^T Q_2 (L\mathcal{G})$.

Using Lemma 2, if there is a constant $\gamma > 0$ such that

$$\begin{aligned}
&\begin{pmatrix} \Theta_{11} & \mathcal{F}\mathcal{A}_d & \mathcal{F}\mathcal{A}_0 - \mathcal{A}^T\mathcal{F} - (L\mathcal{G})^T \mathcal{B}^T \mathcal{F} & \mathcal{A}^T\mathcal{F} + (L\mathcal{G})^T \mathcal{B}^T \mathcal{F} \\ * & -\mathcal{F} & -\mathcal{A}_d{}^T \mathcal{F} & \mathcal{A}_d{}^T \mathcal{F} \\ * & * & -\mathcal{F}\mathcal{A}_0 - \mathcal{A}_0{}^T \mathcal{F} & \mathcal{F} + \mathcal{A}_0{}^T \mathcal{F} \\ * & * & * & -2\mathcal{F} \end{pmatrix} \\
&+ \gamma \begin{pmatrix} \mathcal{F}\mathcal{H} \\ 0 \\ -\mathcal{F}\mathcal{H} \\ \mathcal{F}\mathcal{H} \end{pmatrix} \begin{pmatrix} \mathcal{H}^T\mathcal{F} & 0 & -\mathcal{H}^T\mathcal{F} & \mathcal{H}^T\mathcal{F} \end{pmatrix} \\
&+ \gamma^{-1} \begin{pmatrix} \mathcal{E}_0{}^T + \mathcal{G}^T L^T \mathcal{E}_3{}^T \\ \mathcal{E}_1{}^T \\ \mathcal{E}_2{}^T \\ 0 \end{pmatrix} \begin{pmatrix} \mathcal{E}_0 + \mathcal{E}_3 L\mathcal{G} & \mathcal{E}_1 & \mathcal{E}_2 & 0 \end{pmatrix} \\
&< 0.
\end{aligned} \tag{22}$$

Then $\psi < 0$. Hence, using Lemma 1 twice, yields

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \mathcal{F}\mathcal{A}_d & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & \gamma\mathcal{F}\mathcal{H} & Q_1 & (L\mathcal{G})^T Q_2 \\ * & -\mathcal{F} & -\mathcal{A}_d{}^T\mathcal{F} & \mathcal{A}_d{}^T\mathcal{F} & \mathcal{E}_1{}^T & 0 & 0 & 0 \\ * & * & \Sigma_{33} & \Sigma_{34} & \mathcal{E}_2{}^T & -\gamma\mathcal{F}\mathcal{H} & 0 & 0 \\ * & * & * & -2\mathcal{F} & 0 & \gamma\mathcal{F}\mathcal{H} & 0 & 0 \\ * & * & * & * & -\gamma I & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & * & * & -Q_1 & 0 \\ * & * & * & * & * & * & * & -Q_2 \end{pmatrix} < 0, \tag{23}$$

where $\Sigma_{11} \triangleq \mathcal{F}\mathcal{A} + (L\mathcal{G})^T \mathcal{B}^T \mathcal{F} + \mathcal{A}^T\mathcal{F} + \mathcal{F}\mathcal{B}L\mathcal{G} + \sigma\mathcal{F}$, $\Sigma_{13} \triangleq \mathcal{F}\mathcal{A}_0 - (L\mathcal{G})^T \mathcal{B}^T \mathcal{F} - \mathcal{A}^T\mathcal{F}$, $\Sigma_{14} \triangleq (L\mathcal{G})^T \mathcal{B}^T \mathcal{F} + \mathcal{A}^T\mathcal{F}$, $\Sigma_{15} \triangleq \mathcal{E}_0{}^T + \mathcal{G}^T L^T \mathcal{E}_3{}^T$, $\Sigma_{33} \triangleq -\mathcal{F}\mathcal{A}_0 - \mathcal{A}_0{}^T\mathcal{F}$, and $\Sigma_{34} \triangleq \mathcal{F} + \mathcal{A}_0{}^T\mathcal{F}$.

Pre- and post-multiplying Eq. (23) by $diag\left\{\mathcal{F}^{-1}, \mathcal{F}^{-1}, \mathcal{F}^{-1}, \mathcal{F}^{-1}, I, I, I, I\right\}$ yields

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \mathcal{A}_d\mathcal{F}^{-1} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} & \gamma\mathcal{H} & \mathcal{F}^{-1}Q_1 & \mathcal{F}^{-1}(L\mathcal{G})^T Q_2 \\ * & -\mathcal{F}^{-1} & -\mathcal{F}^{-1}\mathcal{A}_d{}^T & \mathcal{F}^{-1}\mathcal{A}_d{}^T & \mathcal{F}^{-1}\mathcal{E}_1{}^T & 0 & 0 & 0 \\ * & * & \Lambda_{33} & \Lambda_{34} & \mathcal{F}^{-1}\mathcal{E}_2{}^T & -\gamma\mathcal{H} & 0 & 0 \\ * & * & * & -2\mathcal{F}^{-1} & 0 & \gamma\mathcal{H} & 0 & 0 \\ * & * & * & * & -\gamma I & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & * & * & -Q_1 & 0 \\ * & * & * & * & * & * & * & -Q_2 \end{pmatrix} < 0, \tag{24}$$

where $\Lambda_{11} \triangleq \mathcal{A}\mathcal{F}^{-1} + \mathcal{F}^{-1}\mathcal{A}^T + \mathcal{F}^{-1}(L\mathcal{G})^T\mathcal{B}^T + \mathcal{B}L\mathcal{G}\mathcal{F}^{-1} + \mathcal{F}^{-1}\alpha$, $\Lambda_{13} \triangleq \mathcal{A}_0\mathcal{F}^{-1} - \mathcal{F}^{-1}\mathcal{A}^T - \mathcal{F}^{-1}(L\mathcal{G})^T\mathcal{B}^T$, $\Lambda_{14} \triangleq \mathcal{F}^{-1}\mathcal{A}^T + \mathcal{F}^{-1}(L\mathcal{G})^T\mathcal{B}^T$, $\Lambda_{15} \triangleq \mathcal{F}^{-1}{\mathcal{E}_0}^T + \mathcal{F}^{-1}(L\mathcal{G})^T{\mathcal{E}_3}^T$, $\Lambda_{33} \triangleq -\mathcal{A}_0\mathcal{F}^{-1} - \mathcal{F}^{-1}{\mathcal{A}_0}^T$, and $\Lambda_{34} \triangleq \mathcal{F}^{-1} + \mathcal{F}^{-1}{\mathcal{A}_0}^T$. Considering $\alpha > 1$, $\mathcal{X} = L\mathcal{G}\tilde{\mathcal{F}}$, and $\tilde{\mathcal{F}} = \mathcal{F}^{-1}$, we end up the condition (15). Applying Lemma 4, the overall asymptotic stability of (14) is guaranteed. Besides,

$$\begin{aligned} {}^{C}\mathfrak{D}^q\left(\upsilon^T(t)\mathcal{F}\upsilon(t)\right) &\leq -\upsilon^T(t)Q_1\upsilon(t) - \upsilon^T(t)(L\mathcal{G})^TQ_2(L\mathcal{G})\,\upsilon(t) \\ &\leq 0. \end{aligned} \tag{25}$$

Then, we can calculate the GCC value as follows: To this end, integrating (25) with FO q gives

$$V(\mathfrak{n}, \upsilon(\mathfrak{n})) - V(0, \upsilon(0)) \leq -\mathcal{M}. \tag{26}$$

From $V(\mathfrak{n}, \upsilon(\mathfrak{n})) \geq 0$, the subsequent inequality always holds

$$\begin{aligned} \mathcal{M} &\leq V(0, \upsilon(0)) - V(\mathfrak{n}, \upsilon(\mathfrak{n})) \leq V(0, \upsilon(0)) \\ &\leq \lambda_{\max}(\tilde{\mathcal{F}}^{-1})(\|f\|)^2 = \mathcal{M}^*. \end{aligned} \tag{27}$$

This concludes the proof.

3.2. Dynamic output-feedback control solution

Here, the dynamic OFC law is adopted as

$$\begin{aligned} &{}^{C}\mathfrak{D}^q\xi_k(t) = \mathcal{A}_k\xi_k(t) + \mathcal{B}_k y(t), \\ &w(t) = \mathcal{G}_k\xi_k(t), \end{aligned} \tag{28}$$

where $\xi_k(t) \in \mathbb{R}^n_k$ represents the state vector of the controller and the matrices $\mathcal{A}_k, \mathcal{B}_k$, and $\mathcal{G}_k$ (with appropriate dimensions) need to be determined. Therefore, considering system (10) in the light of the controller (28), one can obtain the closed-loop system equations as follows:

$$\begin{aligned} &{}^{C}\mathfrak{D}^q\bar{\xi}\,(t) = (\bar{\mathcal{A}} + \Delta\bar{\mathcal{A}})\bar{\xi}(t) + (\bar{\mathcal{A}}_d + \Delta\bar{\mathcal{A}}_d)\bar{\xi}(t-\varrho) + (\bar{\mathcal{A}}_0 + \Delta\bar{\mathcal{A}}_0){}^{C}\mathfrak{D}^q\bar{\xi}\,(t-\varrho)\,, \\ &\bar{\xi}(t) = f(t), \quad t \in [-\varrho, 0], \end{aligned} \tag{29}$$

where $\bar{\xi}^T(t) = \begin{pmatrix} \xi\,(t) & \xi_k\,(t) \end{pmatrix}$, $\bar{\mathcal{A}} = \begin{pmatrix} \mathcal{A} & \mathcal{B}\mathcal{G}_k \\ \mathcal{B}_k\mathcal{G} & \mathcal{A}_k \end{pmatrix}$, $\bar{\mathcal{A}}_d = \begin{pmatrix} \mathcal{A}_d & 0 \\ 0 & 0 \end{pmatrix}$, $\bar{\mathcal{A}}_0 = \begin{pmatrix} \mathcal{A}_0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Delta\bar{\mathcal{A}} = \begin{pmatrix} \Delta\mathcal{A} & \Delta\mathcal{B}\mathcal{G}_k \\ 0 & 0 \end{pmatrix}$, $\Delta\bar{\mathcal{A}}_0 = \begin{pmatrix} \Delta\mathcal{A}_0 & 0 \\ 0 & 0 \end{pmatrix}$, and $\Delta\bar{\mathcal{A}}_d = \begin{pmatrix} \Delta\mathcal{A}_d & 0 \\ 0 & 0 \end{pmatrix}$.

For the closed-loop system representation (29), let us define the cost function

$$\mathcal{M} = \frac{1}{\Gamma(q)}\int_0^{\mathfrak{n}} (\mathfrak{n} - s)^{q-1}\mathcal{Q}(s)\mathrm{d}s,\ \forall\mathfrak{n} > 0, \tag{30}$$

with $\mathcal{Q}(s) = \xi^T(s)S_1\xi(s) + \xi_k^T(s)S_2\xi_k(s) + w^T(s)S_3w(s)$, where S_1, S_2, S_3 are the positive definite symmetric matrices.

Next theorem provides a robust dynamic OFC solution to the problem stated in Section 2.

Theorem 2: Consider system (29). For given S_1, S_2, and S_3, if one can find a constant $\lambda > 0$, appropriately dimensioned matrices $\mathcal{X}$, Y, W, and $\bar{\mathcal{F}}$ that satisfy

$$\begin{pmatrix} \mathcal{A}\bar{\mathcal{F}} + \bar{\mathcal{F}}\mathcal{A}^T + \bar{\mathcal{F}} & \mathcal{B}\mathcal{X} + Y & \mathcal{A}_d\bar{\mathcal{F}} & \mathcal{A}_0\bar{\mathcal{F}} - \bar{\mathcal{F}}\mathcal{A}^T & \bar{\mathcal{F}}\mathcal{A}^T & \bar{\mathcal{F}}{\mathcal{E}_0}^T & \lambda\mathcal{H} & \bar{\mathcal{F}}S_1 & 0 & 0 \\ * & W + W^T & 0 & -\mathcal{X}^T\mathcal{B}^T & \mathcal{X}^T\mathcal{B}^T & \mathcal{X}^T{\mathcal{E}_3}^T & 0 & 0 & \bar{\mathcal{F}}S_2 & \mathcal{X}^T \\ * & * & -\bar{\mathcal{F}} & -\bar{\mathcal{F}}{\mathcal{A}_d}^T & \bar{\mathcal{F}}{\mathcal{A}_d}^T & \bar{\mathcal{F}}{\mathcal{E}_1}^T & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{\mathcal{F}}{\mathcal{A}_0}^T - \mathcal{A}_0\bar{\mathcal{F}} & \bar{\mathcal{F}} + \bar{\mathcal{F}}{\mathcal{A}_0}^T & \bar{\mathcal{F}}{\mathcal{E}_2}^T & -\lambda\mathcal{H} & 0 & 0 & 0 \\ * & * & * & * & -2\bar{\mathcal{F}} & 0 & \lambda\mathcal{H} & 0 & 0 & 0 \\ * & * & * & * & * & -\lambda I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\lambda I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -S_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & -S_2 & 0 \\ * & * & * & * & * & * & * & * & * & -{S_3}^{-1} \end{pmatrix} < 0, \tag{31}$$

then the closed-loop system (29) is asymptotically stable with the GCC value

$$\mathcal{M}^\star = (\|f\|)^2. \tag{32}$$

and the controller matrices can be recovered as follows

$$\begin{aligned} &\mathcal{A}_k = W\bar{\mathcal{F}}^{-1}, \\ &{\mathcal{B}_k}^T = (CC^T)^{-1}\mathcal{G}\bar{\mathcal{F}}^{-1}Y, \\ &\mathcal{G}_k = \mathcal{X}\bar{\mathcal{F}}^{-1}, \end{aligned} \tag{33}$$

Proof: Choosing the Lyapunov function candidate as

$$V(t) = \bar{\xi}^T(t)\mathcal{F}\bar{\xi}(t). \tag{34}$$

From Lemma 3, one can obtain a bound on the FO derivative of (34) as follows:

$$\begin{aligned} &{}^C\mathfrak{D}^q V(\bar{\xi}(t)) + \bar{\xi}(t)^T(t)S\bar{\xi}(t) + w^T(t)S_3w(t) \\ &\leq (\bar{\xi}(t)^T(t)\mathcal{F})^C\mathfrak{D}^q\bar{\xi}(t) + ({}^C\mathfrak{D}^q\bar{\xi}(t))^T\mathcal{F}\bar{\xi}(t) \\ &\quad + \xi^T(t)S_1\xi(t) + \xi_k^T(t)S_2\xi_k(t) + w^T(t)S_3w(t) \\ &\leq \xi(t)^T(\mathcal{F}\mathcal{A} + S_1 + \mathcal{F}\Delta\mathcal{A} + \Delta\mathcal{A}^T\mathcal{F} + \mathcal{A}^T\mathcal{F})\xi(t) \\ &\quad + \xi^T(t)\,\mathcal{F}\,(\mathcal{A}_d + \Delta\mathcal{A}_d)\xi(t-\varrho) \\ &\quad + \xi^T(t-\varrho)\,({\mathcal{A}_d}^T + \Delta{\mathcal{A}_d}^T)\mathcal{F}\xi(t) \\ &\quad + \xi^T(t)\mathcal{F}(\mathcal{A}_0 + \Delta\mathcal{A}_0)^C\mathfrak{D}^q\xi(t-\varrho) \\ &\quad + {}^C\mathfrak{D}^q\xi^T(t-\varrho)({\mathcal{A}_0}^T + \Delta{\mathcal{A}_0}^T)\mathcal{F}\xi(t) \\ &\quad + {\xi_k}^T(t)(\mathcal{F}\mathcal{B}_k\mathcal{G} + {\mathcal{G}_k}^T\mathcal{B}^T\mathcal{F} + {\mathcal{G}_k}^T\Delta\mathcal{B}^T\mathcal{F})\xi(t) \\ &\quad + \xi^T(t)(\mathcal{F}\Delta\mathcal{B}\mathcal{G}_k + \mathcal{G}^T{\mathcal{B}_k}^T\mathcal{F} + \mathcal{F}\mathcal{B}\mathcal{G}_k)\xi_k(t) \\ &\quad + {\xi_k}^T(t)(\mathcal{F}\mathcal{A}_k + S_2 + {\mathcal{G}_k}^TS_3\mathcal{G}_k + {\mathcal{A}_k}^T\mathcal{F})\xi_k(t). \end{aligned} \tag{35}$$

Considering (34) and using Lemma 4, whenever $\xi(t)$ satisfies

$$V(t+\delta,\xi(t+\delta)) < \mu V(t,\xi(t)),\ -\varrho \le \delta \le 0, \tag{36}$$

for some $\mu > 1$, we get

$$\mu\xi^T(t)\mathcal{F}\xi(t) - \xi^T(t-\varrho)\mathcal{F}\xi(t-\varrho) \ge 0. \tag{37}$$

Combining (35) and (37) and using the following null equation on the right hand side of inequality (35)

$$\begin{aligned} &2\left({}^C\mathfrak{D}^q\upsilon^T(t)\ -{}^C\ \mathfrak{D}^q\upsilon^T(t-\varrho)\right)\mathcal{F}\left(-{}^C\mathfrak{D}^q\upsilon(t) + (\mathcal{A}+\Delta\mathcal{A})\upsilon(t)\right.\\ &\quad\left.+(\mathcal{B}\mathcal{G}_k+\Delta\mathcal{B}\mathcal{G}_k)\upsilon_k(t) + (\mathcal{A}_d+\Delta\mathcal{A}_d)\upsilon(t-\varrho) + (\mathcal{A}_0+\Delta\mathcal{A}_0)^C\mathfrak{D}^q\upsilon(t-\varrho)\right) = 0, \end{aligned}$$

yields

$$\begin{aligned} &{}^C\mathfrak{D}^qV(\bar{\xi}(t)) + \bar{\xi}(t)^T(t)S\bar{\xi}(t) + w^T(t)S_3w(t)\\ &\le (\bar{\xi}(t)^T(t)\mathcal{F})^C\mathfrak{D}^q\bar{\xi}(t) + ({}^C\mathfrak{D}^q\bar{\xi}(t))^T\mathcal{F}\bar{\xi}(t) + \xi^T(t)S_1\xi(t) + \xi_k^T(t)S_2\xi_k(t) + w^T(t)S_3w(t)\\ &\le \xi(t)^T\left(\mathcal{F}\mathcal{A} + \mathcal{A}^T\mathcal{F} + S_1 + \mu\mathcal{F} + \mathcal{F}\Delta\mathcal{A} + \Delta\mathcal{A}^T\mathcal{F}\right)\xi(t) + \xi^T(t)\,\mathcal{F}\,(\mathcal{A}_d+\Delta\mathcal{A}_d)\xi(t-\varrho)\\ &\quad + \xi^T(t-\varrho)\,({\mathcal{A}_d}^T + \Delta{\mathcal{A}_d}^T)\mathcal{F}\xi(t) + \xi^T(t)\mathcal{F}(\mathcal{A}_0+\Delta\mathcal{A}_0)^C\mathfrak{D}^q\xi(t-\varrho)\\ &\quad + {}^C\mathfrak{D}^q\xi^T(t-\varrho)({\mathcal{A}_0}^T + \Delta{\mathcal{A}_0}^T)\mathcal{F}\xi(t) + {\xi_k}^T(t)(\mathcal{F}\mathcal{B}_k\mathcal{G} + {\mathcal{G}_k}^T\Delta\mathcal{B}^T\mathcal{F} + {\mathcal{G}_k}^T\mathcal{B}^T\mathcal{F})\xi(t)\\ &\quad + \xi^T(t)(\mathcal{G}^T{\mathcal{B}_k}^T\mathcal{F} + \mathcal{F}\Delta\mathcal{B}\mathcal{G}_k + \mathcal{F}\mathcal{B}\mathcal{G}_k)\xi_k(t) + {\xi_k}^T(t)({\mathcal{A}_k}^T\mathcal{F} + \mathcal{F}\mathcal{A}_k + S_2 + {\mathcal{G}_k}^TS_3\mathcal{G}_k)\xi_k(t)\\ &\quad - \xi^T(t-\varrho)\,\mathcal{F}\xi(t-\varrho) - 2\,{}^C\mathfrak{D}^q\upsilon^T(t)\mathcal{F}^C\mathfrak{D}^q\upsilon(t) + 2\,{}^C\mathfrak{D}^q\upsilon^T(t)(\mathcal{F}\mathcal{A} + \mathcal{F}\Delta\mathcal{A})\upsilon(t)\\ &\quad + 2\,{}^C\mathfrak{D}^q\upsilon^T(t)(\mathcal{F}\mathcal{B}\mathcal{G}_k + \mathcal{F}\Delta\mathcal{B}\mathcal{G}_k)\upsilon_k(t) + 2\,{}^C\mathfrak{D}^q\upsilon^T(t)(\mathcal{F}\mathcal{A}_d + \mathcal{F}\Delta\mathcal{A}_d)\upsilon(t-\varrho)\\ &\quad + 2\,{}^C\mathfrak{D}^q\upsilon^T(t)(\mathcal{F}\mathcal{A}_0 + \mathcal{F}\Delta\mathcal{A}_0)\,\mathfrak{D}^q\upsilon(t-\varrho) + 2\,{}^C\mathfrak{D}^q\upsilon^T(t-\varrho)\mathcal{F}^C\mathfrak{D}^q\upsilon(t)\\ &\quad - 2\,{}^C\mathfrak{D}^q\upsilon^T(t-\varrho)(\mathcal{F}\mathcal{A} + \mathcal{F}\Delta\mathcal{A})\upsilon(t) - 2\,{}^C\mathfrak{D}^q\upsilon^T(t-\varrho)(\mathcal{F}\mathcal{B}\mathcal{G}_k + \mathcal{F}\Delta\mathcal{B}\mathcal{G}_k)\upsilon(t)\\ &\quad - 2\,{}^C\mathfrak{D}^q\upsilon^T(t-\varrho)(\mathcal{F}\mathcal{A}_d + \mathcal{F}\Delta\mathcal{A}_d)\upsilon(t-\varrho) - 2\,{}^C\mathfrak{D}^q\upsilon^T(t-\varrho)(\mathcal{F}\mathcal{A}_0 + \mathcal{F}\Delta\mathcal{A}_0)^C\mathfrak{D}^q\upsilon(t-\varrho)\\ &\le \psi^T(t)\Pi\psi(t). \end{aligned} \tag{38}$$

Here, $\psi^T(t) \triangleq \Big(\xi^T(t),\quad {\xi_k}^T(t),\quad \xi^T(t-\varrho),\quad {}^C\mathfrak{D}^q\xi^T(t-\varrho),\quad {}^C\mathfrak{D}^q\upsilon^T(t)\Big)$ and

$$\Pi \triangleq \begin{pmatrix} \Pi_{11} & \Pi_{12} & \mathcal{F}(\mathcal{A}_d+\Delta\mathcal{A}_d) & \Pi_{14} & (\mathcal{A}+\Delta\mathcal{A})^T\mathcal{F}\\ * & \Pi_{22} & 0 & \Pi_{24} & \Pi_{25}\\ * & * & -\mathcal{F} & -(\mathcal{A}_d+\Delta\mathcal{A}_d)^T\mathcal{F} & (\mathcal{A}_d+\Delta\mathcal{A}_d)^T\mathcal{F}\\ * & * & * & -\mathcal{F}\mathcal{A}_0 - {\mathcal{A}_0}^T\mathcal{F} & \mathcal{F} + {\mathcal{A}_0}^T\mathcal{F}\\ * & * & * & * & -2\mathcal{F} \end{pmatrix} < 0,$$

where

$$
\begin{aligned}
\Pi_{11} &\triangleq \mathcal{F}\mathcal{A} + \mathcal{A}^T\mathcal{F} + S_1 + \mu\mathcal{F} + \mathcal{F}\Delta\mathcal{A} + \Delta\mathcal{A}^T\mathcal{F}, \\
\Pi_{12} &\triangleq \mathcal{F}\mathcal{B}\mathcal{G}_k + \mathcal{F}\Delta\mathcal{B}\mathcal{G}_k + \mathcal{G}^T{\mathcal{B}_k}^T\mathcal{F}, \\
\Pi_{22} &\triangleq \mathcal{F}\mathcal{A}_k + {\mathcal{A}_k}^T\mathcal{F} + S_2 + {\mathcal{G}_k}^T S_3\mathcal{G}_k, \\
\Pi_{14} &\triangleq \mathcal{F}(\mathcal{A}_0 + \Delta\mathcal{A}_0) - (\mathcal{A} + \Delta\mathcal{A})^T\mathcal{F}, \\
\Pi_{24} &\triangleq -{\mathcal{G}_k}^T\mathcal{B}^T\mathcal{F} - {\mathcal{G}_k}^T\Delta\mathcal{B}^T\mathcal{F}, \\
\Pi_{25} &\triangleq -\Pi_{24}.
\end{aligned} \tag{39}
$$

Note that the above inequality can be decomposed as

$$
\begin{aligned}
&\underbrace{\begin{pmatrix}
\Omega_{11} & \Omega_{12} & \mathcal{F}\mathcal{A}_d & \mathcal{F}\mathcal{A}_0 - \mathcal{A}^T\mathcal{F} & \mathcal{A}^T\mathcal{F} \\
* & \Omega_{22} & 0 & -{\mathcal{G}_k}^T\mathcal{B}^T\mathcal{F} & {\mathcal{G}_k}^T\mathcal{B}^T\mathcal{F} \\
* & * & -\mathcal{F} & -{\mathcal{A}_d}^T\mathcal{F} & {\mathcal{A}_d}^T\mathcal{F} \\
* & * & * & \Omega_{44} & \Omega_{45} \\
* & * & * & * & -2\mathcal{F}
\end{pmatrix}}_{\Omega}
+ \begin{pmatrix} \mathcal{F}\mathcal{H} \\ 0 \\ 0 \\ -\mathcal{F}\mathcal{H} \\ \mathcal{F}\mathcal{H} \end{pmatrix} \aleph(t) \begin{pmatrix} \mathcal{E}_0 & \mathcal{E}_3\mathcal{G}_k & \mathcal{E}_1 & \mathcal{E}_2 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} {\mathcal{E}_0}^T \\ {\mathcal{G}_k}^T{\mathcal{E}_3}^T \\ {\mathcal{E}_1}^T \\ {\mathcal{E}_2}^T \\ 0 \end{pmatrix} \aleph^T(t) \begin{pmatrix} \mathcal{H}^T\mathcal{F} & 0 & 0 & -\mathcal{H}^T\mathcal{F} & \mathcal{H}^T\mathcal{F} \end{pmatrix} < 0,
\end{aligned} \tag{40}
$$

where $\Omega_{11} \triangleq \mathcal{F}\mathcal{A} + \mathcal{A}^T\mathcal{F} + S_1 + \mu\mathcal{F}$, $\Omega_{12} \triangleq \mathcal{F}\mathcal{B}\mathcal{G}_k + \mathcal{G}^T{\mathcal{B}_k}^T\mathcal{F}$, $\Omega_{22} \triangleq {\mathcal{G}_k}^T S_3\mathcal{G}_k + {\mathcal{A}_k}^T\mathcal{F} + \mathcal{F}\mathcal{A}_k + S_2$, $\Omega_{44} \triangleq -\mathcal{F}\mathcal{A}_0 - {\mathcal{A}_0}^T\mathcal{F}$, and $\Omega_{45} \triangleq \mathcal{F} + {\mathcal{A}_0}^T\mathcal{F}$. Then, in the light of Lemma 2, inequality (40) is feasible whenever there exists $\lambda \in \mathbb{R}^+$ satisfying

$$
\begin{aligned}
&\Omega + \lambda \begin{pmatrix} \mathcal{F}\mathcal{H} \\ 0 \\ 0 \\ -\mathcal{F}\mathcal{H} \\ \mathcal{F}\mathcal{H} \end{pmatrix} \begin{pmatrix} \mathcal{H}^T\mathcal{F} & 0 & 0 & -\mathcal{H}^T\mathcal{F} & \mathcal{H}^T\mathcal{F} \end{pmatrix} \\
&+ \lambda^{-1} \begin{pmatrix} {\mathcal{E}_0}^T \\ {\mathcal{G}_k}^T{\mathcal{E}_3}^T \\ {\mathcal{E}_1}^T \\ {\mathcal{E}_2}^T \\ 0 \end{pmatrix} \begin{pmatrix} \mathcal{E}_0 & \mathcal{E}_3\mathcal{G}_k & \mathcal{E}_1 & \mathcal{E}_2 & 0 \end{pmatrix} < 0,
\end{aligned} \tag{41}
$$

From Lemma 1, (41) can be expressed as

$$\begin{pmatrix} \Delta_{11} & \Delta_{12} & \mathcal{F}\mathcal{A}_d & \mathcal{F}\mathcal{A}_0-\mathcal{A}^T\mathcal{F} & \mathcal{A}^T\mathcal{F} & {\mathcal{E}_0}^T & \lambda\mathcal{F}\mathcal{H} & S_1 & 0 & 0 \\ * & \Delta_{22} & 0 & -{\mathcal{G}_k}^T\mathcal{B}^T\mathcal{F} & {\mathcal{G}_k}^T\mathcal{B}^T\mathcal{F} & {\mathcal{G}_k}^T{\mathcal{E}_3}^T & 0 & 0 & S_2 & {\mathcal{G}_k}^T \\ * & * & -\mathcal{F} & -{\mathcal{A}_d}^T\mathcal{F} & {\mathcal{A}_d}^T\mathcal{F} & {\mathcal{E}_1}^T & 0 & 0 & 0 & 0 \\ * & * & * & \Delta_{44} & \Delta_{45} & {\mathcal{E}_2}^T & -\lambda\mathcal{F}\mathcal{H} & 0 & 0 & 0 \\ * & * & * & * & -2\mathcal{F} & 0 & \lambda\mathcal{F}\mathcal{H} & 0 & 0 & 0 \\ * & * & * & * & * & -\lambda I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\lambda I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -S_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & -S_2 & 0 \\ * & * & * & * & * & * & * & * & * & -{S_3}^{-1} \end{pmatrix} < 0, \tag{42}$$

where $\Delta_{11} \triangleq \mathcal{F}\mathcal{A}+\mathcal{A}^T\mathcal{F}+\mu\mathcal{F}$, $\Delta_{12} \triangleq \mathcal{F}\mathcal{B}\mathcal{G}_k+\mathcal{G}^T{\mathcal{B}_k}^T\mathcal{F}$, $\Delta_{22} \triangleq \mathcal{F}\mathcal{A}_k+{\mathcal{A}_k}^T\mathcal{F}$, $\Delta_{44} \triangleq -\mathcal{F}\mathcal{A}_0-{\mathcal{A}_0}^T\mathcal{F}$, and $\Delta_{45} \triangleq \mathcal{F}+{\mathcal{A}_0}^T\mathcal{F}$. Then, pre- and post-multiplying (42) by $diag\left\{\mathcal{F}^{-1},\mathcal{F}^{-1},\mathcal{F}^{-1},\mathcal{F}^{-1},\mathcal{F}^{-1},I,I,I,I,I\right\}$ yields

$$\begin{pmatrix} \Xi_{11} & \Xi_{12} & \mathcal{A}_d\mathcal{F}^{-1} & \Xi_{14} & \mathcal{F}^{-1}\mathcal{A}^T & \mathcal{F}^{-1}{\mathcal{E}_0}^T & \lambda\mathcal{H} & \mathcal{F}^{-1}S_1 & 0 & 0 \\ * & \Xi_{22} & 0 & \Xi_{24} & \Xi_{25} & \Xi_{26} & 0 & 0 & \mathcal{F}^{-1}S_2 & \mathcal{F}^{-1}{\mathcal{G}_k}^T \\ * & * & -\mathcal{F}^{-1} & -\mathcal{F}^{-1}{\mathcal{A}_d}^T & \mathcal{F}^{-1}{\mathcal{A}_d}^T & \mathcal{F}^{-1}{\mathcal{E}_1}^T & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & \Xi_{45} & \mathcal{F}^{-1}{\mathcal{E}_2}^T & -\lambda\mathcal{H} & 0 & 0 & 0 \\ * & * & * & * & -2\mathcal{F}^{-1} & 0 & \lambda\mathcal{H} & 0 & 0 & 0 \\ * & * & * & * & * & -\lambda I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\lambda I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -S_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & -S_2 & 0 \\ * & * & * & * & * & * & * & * & * & -{S_3}^{-1} \end{pmatrix} < 0, \tag{43}$$

where $\Xi_{11} \triangleq \mathcal{A}\mathcal{F}^{-1}+\mathcal{F}^{-1}\mathcal{A}^T+\mathcal{F}^{-1}\mu$, $\Xi_{12} \triangleq \mathcal{B}\mathcal{G}_k\mathcal{F}^{-1}+\mathcal{F}^{-1}\mathcal{G}^T{\mathcal{B}_k}^T$, $\Xi_{14} \triangleq \mathcal{A}_0\mathcal{F}^{-1}-\mathcal{F}^{-1}\mathcal{A}^T$, $\Xi_{22} \triangleq \mathcal{A}_k\mathcal{F}^{-1}+\mathcal{F}^{-1}{\mathcal{A}_k}^T$, $\Xi_{24} \triangleq -\mathcal{F}^{-1}{\mathcal{G}_k}^T\mathcal{B}^T$, $\Xi_{25} \triangleq \mathcal{F}^{-1}{\mathcal{G}_k}^T\mathcal{B}^T$, $\Xi_{26} \triangleq \mathcal{F}^{-1}{\mathcal{G}_k}^T{\mathcal{E}_3}^T$, $\Xi_{44} \triangleq -\mathcal{A}_0\mathcal{F}^{-1}-\mathcal{F}^{-1}{\mathcal{A}_0}^T$, and $\Xi_{45} \triangleq \mathcal{F}^{-1}+\mathcal{F}^{-1}{\mathcal{A}_0}^T$.

Let $\mu > 1$, $\bar{\mathcal{F}} = \mathcal{F}^{-1}$, and define $\mathcal{X} \triangleq \mathcal{G}_k\bar{\mathcal{F}}, Y \triangleq \bar{\mathcal{F}}\mathcal{G}^T{\mathcal{B}_k}^T$ and $W \triangleq \mathcal{A}_k\bar{\mathcal{F}}$. Then, Eq. (43) can be represented as (31). Applying Lemma 4, the asymptotical stability of the overall system (29) with the dynamic OFC law (28) is achieved. Besides,

$$\begin{aligned} -\xi^T(t)S_1\xi(t)-\xi_k^T(t)S_2\xi_k(t)-w(t)^TS_3w(t) &\geq {}^C\mathfrak{D}^q\left(\bar{\xi}^T(t)\mathcal{F}\bar{\xi}(t)\right) \\ &\geq 0. \end{aligned} \tag{44}$$

Then, we are ready to calculate the GCC value. Integrating (44) with FO q gives

$$V(\mathfrak{n},\bar{\xi}(\mathfrak{n}))-V(0,\bar{\xi}(0)) \leq -\mathcal{M}. \tag{45}$$

From $V(\mathfrak{n}, \bar{\xi}(\mathfrak{n})) \geq 0$, we conclude that

$$\begin{aligned} \mathcal{M} &\leq V(0, \bar{\xi}(0)) - V(\mathfrak{n}, \bar{\xi}(\mathfrak{n})) \\ &\leq V(0, \bar{\xi}(0)) \\ &\leq (\|f\|)^2 = \mathcal{M}^\star. \end{aligned} \tag{46}$$

This concludes the proof.

Remark 2: As compared to related works, the following advantages of the proposed method can be highlighted:

- In contrast to the existing control designs [38,43,46,52,54], in which the stability is only considered for the closed-loop system, in the current work, by applying GCC technique, we also guarantee an adequate level of the system performance to improve its behavior.
- Most reported works in the literature have considered a state-feedback controller design [40,41,39]. However, it is possible that all the system's states are not accessible. This issue can be sorted out using the output-feedback control technique that we considered in this paper to overcome this limitation. Indeed, in most cases, output variables can be measured directly and possess a clear physical meaning, making output feedback easier to implement than state-feedback control.
- Neutral systems are a kind of important time-delay systems that exist in many engineering applications. So far, there have been very few research results on the FO neutral systems. In addition, the GCC design for such systems has not yet been discussed [38,39,41,43,44,45,52]. Moreover, different from these works, we study the stabilization of FO neutral delay system using the GCC technique.

Remark 3: Theorems 1 and 2 provide sufficient conditions in terms of the LMIs. Using the LMI toolbox in Matlab, or some other free softwares such as SEDUMI and YALMIP, we can easily solve these LMI conditions.

4. Simulation Results

The viability of the developed ideas are verified using two numerical examples, where the modified Adams–Bashforth–Moulton algorithm [60] is employed to solve the FO differential equations.

Example 1: Consider a FO system (10) with parameters

$$\begin{aligned} &\mathcal{A} = \begin{pmatrix} -1 & 0.5 \\ 0 & 1 \end{pmatrix},\ \mathcal{A}_d = \begin{pmatrix} -2 & -0.5 \\ -0.5 & -2.5 \end{pmatrix},\ \mathcal{A}_0 = \begin{pmatrix} 0.2 & 0.1 \\ 0 & 0.2 \end{pmatrix}, \\ &\mathcal{B} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T,\ \mathcal{G} = \begin{pmatrix} 1 & 0 \end{pmatrix},\ \mathcal{H} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T,\ \mathcal{E}_0 = \begin{pmatrix} -0.5 & 0.1 \end{pmatrix}, \\ &\mathcal{E}_1 = \begin{pmatrix} -0.5 & 0.2 \end{pmatrix},\ \mathcal{E}_2 = \begin{pmatrix} -0.3 & 0.2 \end{pmatrix},\ \mathcal{E}_3 = -0.3,\ \varrho = 0.2. \end{aligned}$$

The system response without the controller is shown in Figure 1, that is clearly unstable. The main results stated in Theorem 1 are used to determine static OFC gain. Choosing $Q_1 = \begin{pmatrix} 2.5 & 0 \\ 0 & 1.5 \end{pmatrix}$ and $Q_2 = 2$, one gets:

$$\tilde{\mathcal{F}} = \begin{pmatrix} 0.2424 & -0.0433 \\ -0.0433 & 0.1609 \end{pmatrix},\ \mathcal{X} = \begin{pmatrix} 0.0195 & -0.2758 \end{pmatrix},\ \gamma = 0.1382.$$

The OFC gain $L = -0.2372$ is obtained with $\mathcal{M}^{\star} = 3.5165$.

The achieved results for FO orders $q = 0.9$ and $q = 0.7$ are displayed in Figs. 2–5. Figures 2 and 3 illustrate the closed-loop system response, while the control signals' variations are depicted in Figures 4 and 5. Referring to these results, the state trajectories converge to the origin asymptotically. Moreover, decreasing the values of q results in larger settling times. In summary, the satisfactory performance of the controller is verified.

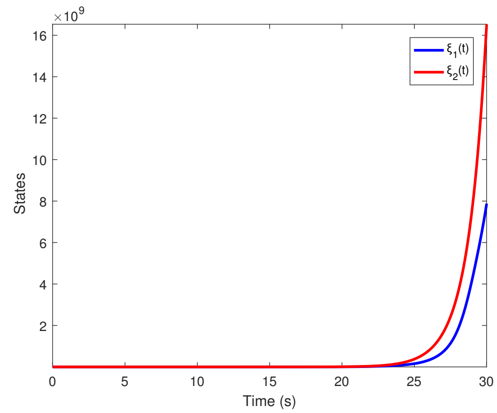

Figure 1: Time response of Example 1 without controller ($q = 0.9$).

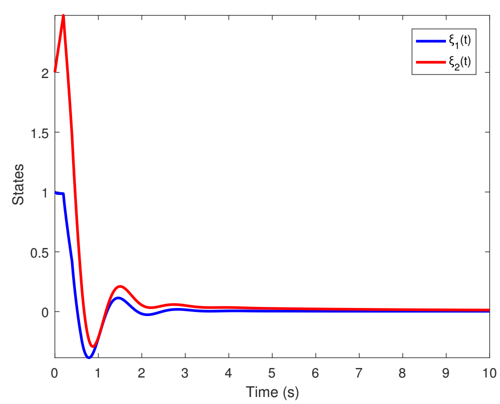

Figure 2: Time response of Example 1 using the proposed controller ($q = 0.9$).

Example 2: In this example, we adopt the two-stage chemical reactor to discusses how the proposed methodology can be related to a specific application. The FO model of reactor system [61] is given by

$$\begin{aligned} \upsilon_1 {}_0^C\mathfrak{D}^q c_1(t) &= f_1\gamma_{1f}(t) + r\gamma_2(t-\mathfrak{n}) + f_d\gamma_d(t) - \upsilon_1(\kappa_1 + \Delta\kappa_1(t))\gamma_1(t) \\ &\quad - (f_1 + r + f_d)\gamma_1(t), \\ \upsilon_2 {}_0^C\mathfrak{D}^q \gamma_2(t) &= (f_1 + r + f_d - f_{p1})\gamma_1(t) + f_2\gamma_{2f}(t) - \upsilon_2(\kappa_2 + \Delta\kappa_2(t))\gamma_2(t) \\ &\quad - (f_{p2} + r)\gamma_2(t), \end{aligned} \tag{47}$$

where f_1 and f_2 represent the feed rates, γ_{1f} and γ_{2f} denote the reactors' feed composition and f_d is the process disturbance corresponding to an extra feed stream with a composition γ_d. Furthermore, the recycle flow rate is

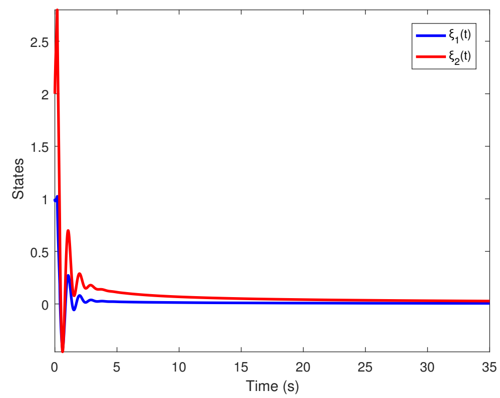

Figure 3: Time response of Example 1 using the proposed controller ($q = 0.7$).

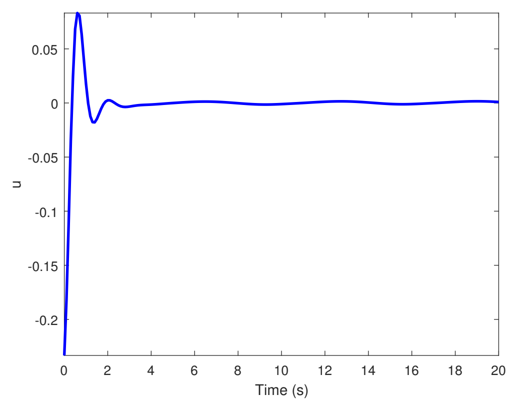

Figure 4: Control signal of Example 1 by employing the proposed controller ($q = 0.9$).

denoted by r, the reactors volumes is represented by υ_1 and υ_2, and $\Delta\kappa_1$ and $\Delta\kappa_2$ stand for the system uncertainties, which are time-varying. Although the parameters are unknown in practice, the upper bounds on their values can be assumed. We define the reactor residence times θ_1 and θ_2 as follows:

$$\theta_1 \triangleq \frac{\upsilon_1}{f_1 + r + f_d} \ , \ \theta_2 \triangleq \frac{\upsilon_2}{f_{p2} + r}. \tag{48}$$

Then, the state-space representation of (47) can be written as

$$^{C}\mathfrak{D}^{q}\upsilon\,(t) = (\mathcal{A} + \Delta\mathcal{A})\upsilon(t) + (\mathcal{A}_d + \Delta\mathcal{A}_d)\upsilon(t - \varrho) + (\mathcal{B} + \Delta\mathcal{B})w\,(t)\,, \tag{49}$$

where

$$\mathcal{A} = \begin{bmatrix} -(\frac{1}{\theta_1} + \kappa_1) & 0 \\ \frac{r_{p2} - r_2 + r}{\upsilon_2} & -(\frac{1}{\theta_2} + \kappa_2) \end{bmatrix}, \mathcal{A}_d(t) = \begin{bmatrix} 0 & \frac{r}{\upsilon_1} \\ 0 & 0 \end{bmatrix}, \mathcal{B} = \begin{pmatrix} \frac{f_1}{\upsilon_1} & 0 \\ 0 & \frac{f_2}{\upsilon_2} \end{pmatrix}, \tag{50}$$

which is a particular case of (14) with $\mathcal{A}_0 = 0$. We set the parameters $\upsilon_1 = \upsilon_2 = 1, f_{p1} = f_{p2} = 0.5, f_1 = 0.4, f_2 = 0.5, r = 0.25, f_d = 0.1, \kappa_1 = \kappa_2 = 1$, and $\theta_2 = 0.5$ as in [61]. Then, $\mathcal{G} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \mathcal{H} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \mathcal{E}_0 =$

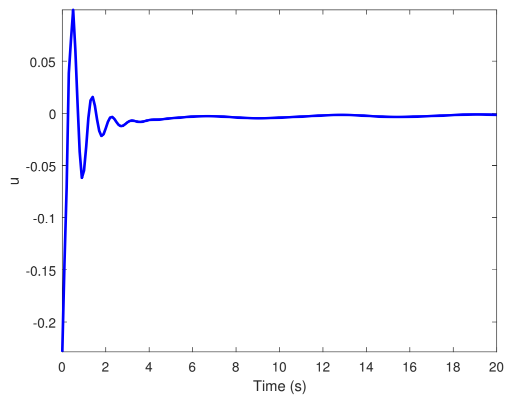

Figure 5: Control signal of Example 1 by employing the proposed controller ($q = 0.7$).

$\begin{pmatrix} -0.6 & -0.6 \end{pmatrix}, \mathcal{E}_1 = \begin{pmatrix} -0.3 & 0.2 \end{pmatrix}, \mathcal{E}_3 = \begin{pmatrix} 0.3 & 0.1 \end{pmatrix}$. Choosing $Q_1 = I, Q_2 = 0.5, \varrho = 1$, yields the following results

$$\tilde{\mathcal{F}} = \begin{pmatrix} 0.4608 & 0.0046 \\ 0.0046 & 0.4185 \end{pmatrix}, \ \mathcal{X} = \begin{pmatrix} 0.5614 & 0.0856 \\ -0.1315 & 0.4185 \end{pmatrix}, \ \gamma = 0.5334.$$

We obtain the OFC gain $L = \begin{pmatrix} 1.2167 & -0.2939 \end{pmatrix}^T$ and the GGC value $\mathcal{M}^\star = 10.87$. The system response and the control signal with $q = 0.9$ and $q = 0.7$ are represented in Figs. 6–9. From these figures, it can be deduced that the state trajectories converge to the origin. Besides, similar to the previous results, decreasing the values of q results in larger settling times. According to the results, we can infer that the system's behavior is satisfactory.

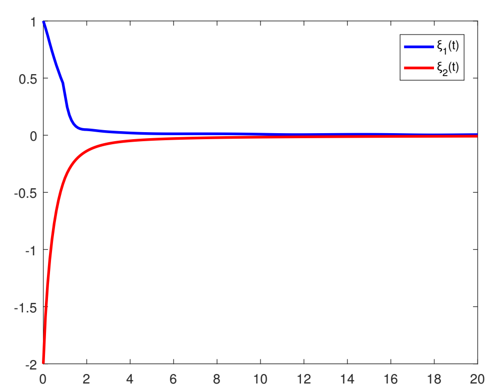

Figure 6: Time response of Example 2 using the proposed controller ($q = 0.9$).

5. Conclusions

The output feedback control problem of uncertain FO neutral-type delay systems was studied in this paper. The Lyapunov theory was used to derive the delay independent conditions for the guaranteed cost stability of such systems in the presence of time-varying parametric uncertainty. Accordingly, dynamic and static feedback control

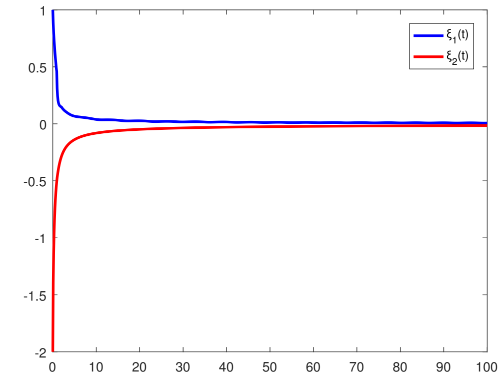

Figure 7: Time response of Example 2 using the proposed controller ($q = 0.7$).

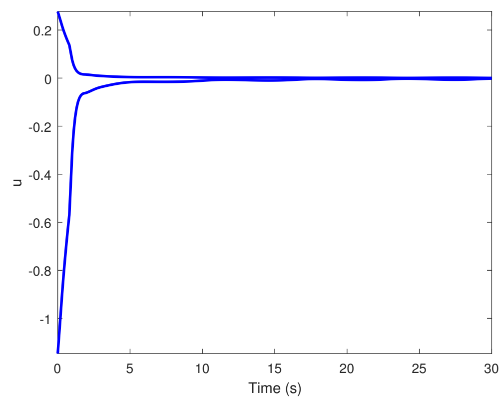

Figure 8: Control signal of Example 2 by employing the proposed controller for $q = 0.9$.

algorithms have been proposed to undertake the problem, while guaranteeing the asymptotic stability of the system. The desirable control performance was confirmed through several simulations. As a future work, it would be useful to study the delay-dependent stability analysis and control of the FO neutral systems.

References

[1] K. Gu, J. Chen, V. L. Kharitonov, Stability of time-delay systems, Springer Science & Business Media, 2003.

[2] M. Shafiya, G. Nagamani, D. Dafik, Global synchronization of uncertain fractional-order BAM neural networks with time delay via improved fractional-order integral inequality, Mathematics and Computers in Simulation 191 (2022) 168–186.

[3] A. Bakhshi, A. Alfi, H. A. Talebi, A. A. Souratgar, M. Yousefi, Stability analysis and performance evaluation of delayed bilateral telerobotic systems over a lossy communication channel, Journal of Systems Science and Complexity 34 (1) (2021) 157–179.

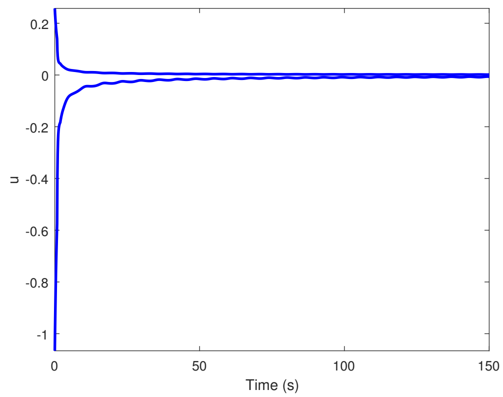

Figure 9: Control signal of Example 2 by employing the proposed controller for $q = 0.7$.

[4] J. Wang, Y. Zhu, Y. Gu, Z. Lu, Solutions of linear uncertain fractional order neutral differential equations, Applied Mathematics and Computation 407 (2021) 126323.

[5] Q.-L. Han, Stability analysis for a partial element equivalent circuit (PEEC) model of neutral type, International Journal of Circuit Theory and Applications 33 (4) (2005) 321–332.

[6] Y. Liu, J. H. Park, F. Fang, On criteria for stability of uncertain Lur'e systems of neutral type, Nonlinear Dynamics 98 (3) (2019) 2185–2194.

[7] F. Du, J.-G. Lu, Finite-time stability of neutral fractional order time delay systems with lipschitz nonlinearities, Applied Mathematics and Computation 375 (2020) 125079.

[8] L. Ucun, I. B. Küçükdemiral, Robust delay-dependent feedforward control of neutral time-delay systems via dynamic IQCs, International Journal of Systems Science 45 (5) (2014) 858–872.

[9] J. A. T. Machado, B. P. Moghaddam, A robust algorithm for nonlinear variable-order fractional control systems with delay, International Journal of Nonlinear Sciences and Numerical Simulation 19 (3-4) (2018) 231–238.

[10] E. S. Alaviyan Shahri, A. Alfi, J. Tenreiro Machado, Robust stability and stabilization of uncertain fractional order systems subject to input saturation, Journal of Vibration and Control 24 (16) (2018) 3676–3683.

[11] E. S. A. Shahri, A. Alfi, J. T. Machado, Lyapunov method for the stability analysis of uncertain fractional-order systems under input saturation, Applied Mathematical Modelling 81 (2020) 663–672.

[12] S. M. A. Pahnehkolaei, A. Alfi, J. T. Machado, Delay independent robust stability analysis of delayed fractional quaternion-valued leaky integrator echo state neural networks with quad condition, Applied Mathematics and Computation 359 (2019) 278–293.

[13] S. Chang, T. Peng, Adaptive guaranteed cost control of systems with uncertain parameters, IEEE Transactions on Automatic Control 17 (4) (1972) 474–483.

[14] J. H. Park, K. Choi, Guaranteed cost control of uncertain nonlinear neutral systems via memory state feedback, Chaos, Solitons & Fractals 24 (1) (2005) 183–190.

[15] L. Mohammadi, A. Alfi, Guaranteed cost control in delayed teleoperation systems under actuator saturation, Iranian Journal of Science and Technology, Transactions of Electrical Engineering 43 (4) (2019) 827–835.

[16] T. H. Lee, J. H. Park, D. Ji, O. Kwon, S.-M. Lee, Guaranteed cost synchronization of a complex dynamical network via dynamic feedback control, Applied Mathematics and Computation 218 (11) (2012) 6469–6481.

[17] B. Shen, Z. Wang, H. Tan, Guaranteed cost control for uncertain nonlinear systems with mixed time-delays: The discrete-time case, European Journal of Control 40 (2018) 62–67.

[18] X. Yang, Y. Wang, X. Zhang, Lyapunov matrix-based method to guaranteed cost control for a class of delayed continuous-time nonlinear systems, IEEE Transactions on Systems, Man, and Cybernetics: Systems, DOI: 10.1109/TSMC.2020.3003545 (2020).

[19] H. He, L. Yan, J. Tu, Guaranteed cost stabilization of time-varying delay cellular neural networks via riccati inequality approach, Neural processing letters 35 (2) (2012) 151–158.

[20] H. He, L. Yan, J. Tu, Guaranteed cost stabilization of cellular neural networks with time-varying delay, Asian Journal of Control 15 (4) (2013) 1224–1227.

[21] H. He, W. Xu, M. Jiang, Guaranteed cost anti-windup stabilization of discrete delayed cellular neural networks, Neural Processing Letters 46 (1) (2017) 343–354.

[22] P. Niamsup, V. N. Phat, A new result on finite-time control of singular linear time-delay systems, Applied Mathematics Letters 60 (2016) 1–7.

[23] C. Lien, Guaranteed cost observer–based controls for a class of uncertain neutral time-delay systems, Journal of Optimization Theory and Applications 126 (1) (2005) 137–156.

[24] V. E. Tarasov, Handbook of Fractional Calculus with Applications, Vol. 5, de Gruyter Berlin, Germany, 2019.

[25] C. Ding, J. Cao, Y. Chen, Fractional-order model and experimental verification for broadband hysteresis in piezoelectric actuators, Nonlinear Dynamics 98 (4) (2019) 3143–3153.

[26] D. Yousri, S. Mirjalili, J. T. Machado, S. B. Thanikanti, A. Fathy, et al., Efficient fractional-order modified harris hawks optimizer for proton exchange membrane fuel cell modeling, Engineering Applications of Artificial Intelligence 100 (2021) 104193.

[27] Y. Mousavi, G. Bevan, I. B. Kucukdemiral, Fault-tolerant optimal pitch control of wind turbines using dynamic weighted parallel firefly algorithm, ISA Transactions, DOI: 10.1016/j.isatra.2021.10.019 (2021).

[28] R. Hilfer, Applications of fractional calculus in physics, World scientific, 2000.

[29] J.-G. Liu, X.-J. Yang, Y.-Y. Feng, P. Cui, On the (N+1)-dimensional local fractional reduced differential transform method and its applications, Mathematical Methods in the Applied Sciences 43 (15) (2020) 8856–8866.

[30] M. Zubair, M. J. Mughal, Q. A. Naqvi, Electromagnetic wave propagation in fractional space, in: Electromagnetic fields and waves in fractional dimensional space, Springer, 2012, pp. 27–60.

[31] B. Xiao, J. Luo, X. Bi, W. Li, B. Chen, Fractional discrete tchebyshev moments and their applications in image encryption and watermarking, Information Sciences 516 (2020) 545–559.

[32] C. J. Silva, D. F. Torres, Stability of a fractional HIV/AIDS model, Mathematics and Computers in Simulation 164 (2019) 180–190.

[33] K. M. Owolabi, High-dimensional spatial patterns in fractional reaction-diffusion system arising in biology, Chaos, Solitons & Fractals 134 (2020) 109723.

[34] J. Shi, K. He, H. Fang, Chaos, Hopf bifurcation and control of a fractional-order delay financial system, Mathematics and Computers in Simulation 194 (2022) 348–364.

[35] I. A. Boulham, A. Boubakir, S. Labiod, L_1 adaptive controller design for a class of fractional order uncertain systems, Mathematics and Computers in Simulation 193 (2022) 232–249.

[36] N. Padmaja, P. Balasubramaniam, Mixed H_∞/passivity based stability analysis of fractional-order gene regulatory networks with variable delays, Mathematics and Computers in Simulation 192 (2022) 167–181.

[37] J. Jia, X. Huang, Y. Li, J. Cao, A. Alsaedi, Global stabilization of fractional-order memristor-based neural networks with time delay, IEEE Transactions on Neural Networks and Learning Systems 31 (3) (2019) 997–1009.

[38] Y. Guo, C. Lin, B. Chen, Q.-G. Wang, Stabilization for singular fractional-order systems via static output feedback, IEEE Access 6 (2018) 71678–71684.

[39] M. V. Thuan, D. C. Huong, Robust guaranteed cost control for time-delay fractional-order neural networks systems, Optimal Control Applications and Methods 40 (4) (2019) 613–625.

[40] L. Chen, R. Wu, L. Yuan, L. Yin, Y. Chen, S. Xu, Guaranteed cost control of fractional-order linear uncertain systems with time-varying delay, Optimal Control Applications and Methods 42 (4) (2021) 1102–1118.

[41] F. Qi, Y. Chai, L. Chen, J. A. Tenreiro Machado, Delay-dependent and order-dependent guaranteed cost control for uncertain fractional-order delayed linear systems, Mathematics 9 (1) (2021) 41.

[42] Y. Fan, X. Huang, Z. Wang, J. Xia, H. Shen, Discontinuous event-triggered control for local stabilization of memristive neural networks with actuator saturation: Discrete-and continuous-time lyapunov methods, IEEE Transactions on Neural Networks and Learning Systems, DOI: 10.1109/TNNLS.2021.3105731 (2021).

[43] M. V. Thuan, D. C. Huong, New results on stabilization of fractional-order nonlinear systems via an LMI approach, Asian Journal of Control 20 (4) (2018) 1541–1550.

[44] R. Zhang, D. Zeng, X. Liu, S. Zhong, Q. Zhong, Improved results on state feedback stabilization for a networked control system with additive time-varying delay components' controller, ISA Transactions 75 (2018) 1–14.

[45] J. Zhang, D. Liu, Y. Ma, P. Yu, Non-fragile H_∞ memory sampled-data state-feedback control for continuous-time nonlinear markovian jump fuzzy systems with time-varying delay, Information Sciences 577 (2021) 214–233.

[46] N. T. Thanh, P. Niamsup, V. N. Phat, Observer-based finite-time control of linear fractional-order systems with interval time-varying delay, International Journal of Systems Science 52 (7) (2021) 1386–1395.

[47] Z. S. Aghayan, A. Alfi, J. Tenreiro Machado, Stability analysis of fractional order neutral-type systems considering time varying delays, nonlinear perturbations, and input saturation, Mathematical Methods in the Applied Sciences 43 (17) (2020) 10332–10345.

[48] Z. S. Aghayan, A. Alfi, J. T. Machado, Robust stability of uncertain fractional order systems of neutral type with distributed delays and control input saturation, ISA Transactions 111 (2021) 144–155.

[49] Z. S. Aghayan, A. Alfi, J. T. Machado, LMI-based stability analysis of fractional order systems of neutral type with time varying delays under actuator saturation, Computational and Applied Mathematics 40 (4) (2021) 1–24.

[50] Z. S. Aghayan, A. Alfi, J. A. Tenreiro Machado, Observer-based control approach for fractional-order delay systems of neutral type with saturating actuator, Mathematical Methods in the Applied Sciences (2021).

[51] L. Tan, C. Li, X. He, T. Huang, Distributed output feedback leader-following consensus for nonlinear multiagent systems with time delay, Nonlinear Dynamics 105 (2) (2021) 1673–1687.

[52] M. V. Thuan, V. N. Phat, H. M. Trinh, Dynamic output feedback guaranteed cost control for linear systems with interval time-varying delays in states and outputs, Applied Mathematics and Computation 218 (21) (2012) 10697–10707.

[53] X. Song, Z. Wang, Dynamic output feedback control for fractional-order systems, Asian Journal of Control 15 (3) (2013) 834–848.

[54] P. Badri, M. Sojoodi, Robust stabilisation of fractional-order interval systems via dynamic output feedback: an lmi approach, International Journal of Systems Science 50 (9) (2019) 1718–1730.

[55] A. Kilbas, Theory and applications of fractional differential equations.

[56] F. Zhang, The Schur complement and its applications, Vol. 4, Springer Science & Business Media, 2006.

[57] L. Xie, Output feedback H_∞ control of systems with parameter uncertainty, International Journal of control 63 (4) (1996) 741–750.

[58] P. Li, L. Chen, R. Wu, J. T. Machado, A. M. Lopes, L. Yuan, Robust asymptotic stability of interval fractional-order nonlinear systems with time-delay, Journal of the Franklin Institute 355 (15) (2018) 7749–7763.

[59] B. Chen, J. Chen, Razumikhin-type stability theorems for functional fractional-order differential systems and applications, Applied Mathematics and Computation 254 (2015) 63–69.

[60] K. Diethelm, N. J. Ford, Analysis of fractional differential equations, Journal of Mathematical Analysis and Applications 265 (2) (2002) 229–248.

[61] S. Phoojaruenchanachai, K. Uahchinkul, Y. Prempraneerach, Robust stabilisation of a state delayed system, IEE Proceedings-Control Theory and Applications 145 (1) (1998) 87–91.