The effect of risk aversion on the optimal project resource rate

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The effect of risk aversion on the optimal project resource rate

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Summary
Under resourcing a project increases the probability of a time overrun. Consequently, project contracts should be designed to encourage an appropriate allocation of resources to the project. A common way to encourage timely completion is to use contracts with time penalties and incentives linked to the completion time. If there are a number of competing contractors, then the project manager can employ a take it or leave it approach in designing the contract. However, where there are very few possible contractors, then a bargaining approach is more appropriate for the contract's construction. Therefore, this paper investigates how close the resource rate stemming from the Nash bargaining contract is to the optimal rate. Risk neutral and risk averse project managers and contractors are considered. It is found that when the contractor is risk neutral, the chosen resource rate is independent of the project completion time distribution no matter whether the project manager is risk neutral or risk averse, and it coincides with the optimal, i.e. centrally coordinated, rate. When the contractor is risk averse, the resource rate is dependent on the project completion time distribution. However, the results indicate that if the contractor is less risk averse than the project manager, then the resource rate is approximately the optimal one. Hence a time based contract designed using Nash bargaining is particularly suitable when the number of possible contractors is small and they are large enough with regard to the project size to be less risk averse than the project manager.

Keywords
Game theory; project completion; resource rate; contract design.

Highlights
• Bargaining resource rate is optimally efficient when the contractor is risk neutral
• An inefficient resource rate occurs when the contractor is the more risk averse party
• Contracts where the contractor takes all the risk given an optimal resource rate
• However, these contracts give lower utilities for the participants
1 Introduction

Project completion times are often significantly late (Love et al. 2013). While there are many factors that can cause the late delivery of a project, the employment of too low a level of resource on the project can be a contributory factor (Moore 2009, Shenhar et al. 2016). The fact that there is often a substantial increase in the resources employed on a project as the target date for its handover nears, suggests that some projects choose too low a resource rate (as increasing resources to crash tasks in PERT (Laslo & Gurevich (2011)) is a standard way to get slipping projects back on track). However, while choosing an inappropriately high resource rate will lower the risk of completion time problems, it will normally increase the contractor’s cost as it is unlikely that the extra resource can be used as efficiently as the resources in the case where the resource rate was lower. Additionally, the extra resource will usually be costlier – this is part of the Project Management Triangle, see for example Cunningham & Thissen (2013).

If the project manager and the contractor are the same company, then the reduced benefits stemming from a later completion time can be traded off against the lower costs from using a lower resource rate in order to arrive at an optimal resource rate. In the literature this centrally coordinated resource rate has been labelled the “first best solution” (Kwon et al. 2010). This paper investigates whether contracts with a linear penalty for late completion can be used to encourage the use of an optimal resource rate by the contractor. This resource rate is desirable as it maximises the net benefit of the project. The larger the benefit and the more fairly that it is split between the project manager and the contractor, the more sustainable their relationship and the greater the chance that they will work together on future projects.

Previous work has considered how to use the Nash bargaining approach to design cost based contracts between the project manager and the contractor (Lippman et al. 2013). However, the project manager has no direct control over the costs the contractor incurs, and so there can be concerns about these costs. Therefore, project managers generally prefer time based contracts as the completion time is easier to determine and it links directly to the project manager’s key target. Here previous work (Kwon et al. 2010) has concentrated on a game theoretic approach based around Nash equilibria and a Stackelberg take it or leave it game controlled by the project manager. The outcome of this work concludes that the contractor is paid the bare minimum while the project manager takes the major part of the profit. This may be appropriate where there are many potential contractors, but where there are very few potential contractors it is inappropriate and it is not conducive to the building up of a long term relationship between a project manager and a contractor. Therefore, this paper investigates how to design time based contracts based on Nash bargaining. It considers the four cases arising from risk neutral and risk averse project managers, and risk neutral and risk averse contractors. The main findings are that appropriate
time based contracts lead to an optimal resource rate being chosen in the case of a risk neutral contractor irrespective of whether the project manager is risk neutral or risk averse, and that less beneficial resource rate choices occur when the risk aversion level of the contractor is greater than the project manager’s risk aversion level. Hence, it is beneficial in terms of the resource rate efficiency if the project manager is more risk averse than the contractor.

The next section discusses the background literature. After this, Section 3 describes the cost models for the project manager and the contractor. Section 4 analyses the four combinations of risk averse and risk neutral parties, before a sensitivity analysis is carried out in Section 5. Section 6 analyses the implications of the research for two types of highways contracts that are used in the USA. The paper concludes by discussing the results and their implications.

2 Background literature

The problem considered is where a project manager has employed a contractor to deliver a project. In these circumstances, the late completion of projects is a frequent experience with Love et al. (2013) reporting overrun rates of 86%, 87% and 88% with the size of average overruns varying from study to study but typically being around 30% to 35% of the agreed target time. Consequently, many contracts for the delivery of projects contain penalties for late delivery and possibly bonuses for expediting early delivery (Arditi et al. 1997) with 42% of projects surveyed by Meng and Gallagher (2012) incorporating a time penalty (Chen and Lee 2017). 72% of the projects with a time penalty were completed on time compared with only 20% of those with no time penalty. However, completing a project in a shorter time usually involves a higher cost. This problem is known as the time-cost trade off problem (De et al. 1995) and is related to the Project Management Triangle trade off. The increased costs can stem from, amongst other things, the use of multiple shifts and overtime, overcrowded workspaces and the hiring of more equipment (Feng et al. 2000, Shr and Chen 2003). Arditi et al. (1997) gives the three key measures of success of a completed project as time, cost and quality, i.e. the corners of the Project Management Triangle. One of the desirable properties of time based contracts is the direct linkage of two of these – time and cost. An additional desirable property of these contracts is that they measure something that can easily be observed, i.e. the completion time. This contrasts with contracts based on the effort level employed (Qi 2018) or on the contractor’s costs (Lippman et al. 2013).

Designing an appropriate time based incentive contract can lead to significant welfare gains – the example in Lewis and Bajari (2014) gives a benefit of 22.5% of the contract value. Shr and Chen (2003) modelled the optimal bid price for a contractor where the project manager has specified the incentive / disincentive contract that will be used, i.e. given the specified time incentives / disincentives and the contractor’s time-cost function, what should be the contractor’s target
completion time? However, as improving the welfare gain is likely to strengthen partner relationships due to the increased welfare being split amongst them, designing contracts that lead to higher welfare levels are important. Therefore, Kwon et al. (2010) considered the maximum level of the welfare benefit that could be achieved by choosing a resource rate for the contractor, i.e. a centrally coordinated resource rate, and then investigated whether a time based take it or leave it contract offered by the project manager could achieve this resource rate when the problem was considered as a Nash equilibrium of a Stackelberg game. The main drawback of this approach is that the contract favours the project manager and the contractor ends up with a very low level of utility. Where the pool of potential contractors is small, this is unlikely to lead to the development of the partnerships with suppliers advocated in de Araujo et al. (2017). In this case, a cooperative Nash bargaining solution approach (Muthoo 1999) rather than a non-cooperative game theoretic approach would seem to be more sensible. Lippman et al. (2013) described such a cooperative approach for cost based contracts, but time based incentive contracts are preferred in practice (Arditi et al. (1997); Shr and Chen (2003); Estevez-Fernandez (2012); Lewis and Bajari (2014); Kerkhove and Vanhoucke (2016); Yang et al. (2016); Chen and Lee (2017)). While the project manager has generally been modelled as being risk neutral in these papers, there has been a difference when modelling the contractor. Kerkhove and Vanhoucke (2016) concentrate on risk neutral contractors as “we are considering economic actors rather than individuals” (page 93), but Lippman et al. (2013), Yang et al. (2016) and Qi (2018) focus on risk averse contractors. Consequently, the issue of how the welfare gains from using time based contracts using Nash bargaining compares with the (optimal) welfare gain from central coordination is an open question. This issue is the focus of this paper with particular emphasis being placed on the consequences of the project manager and the contractor being risk averse or risk neutral.

3 Model

Following the approach in Kwon et al. (2010), let the contract payment from the project manager to the contractor on the completion of the project be

\[
\text{Payment: } g - h T
\]  

(1)

where

\[
T
\]

is the observed project completion time

\[
g
\]

is a constant specified by the project manager

\[
h
\]

is a constant specified by the project manager

The contractor’s cost for a time period of length \( t \) is given by

\[
\text{Cost: } k \lambda^0 t
\]  

(2)
where
\[ n \geq 1 \] models the higher marginal cost (e.g. overtime payments) of employing increased resources mentioned in Section 2. This approach has been adopted in a number of papers including Kwon et al. (2010), Yang et al. (2016) and Qi (2018). In line with these papers, \( n \) was set to the value 2. The sensitivity of the model to the value of \( n \) is investigated using \( n=1.4 \) in Section 5.

\( \lambda \) is the resource rate employed by the contractor.

\( k \) is a constant with units of $ time\(^{-1}\).

Table 1 in Herbsman et al. (1995) estimates the completion time of a $20,000,000 project as 700 days. Assuming a 7 day working week and defining the resource rate to be 1 for this case, i.e. the resource rate used in practice, then
\[ k = \frac{20,000,000}{100} \approx 200,000 \text{ per week}. \]

Therefore, the values used in this paper were \( \lambda=1 \) and \( k=200,000 \). Note that 1 is only the starting value of \( \lambda \), with the model adjusting \( \lambda \) to find an optimal solution. The sensitivity of the model to \( k \) is analysed in Section 5.

The value of the project to the project manager on completion is modelled as
\[ q = q_0 (1 - \omega T) \] (3)

where
\[ \omega \] is the proportion of the project value lost for every extra day (or week) it takes to complete the project. For example, Herbsman et al. (1995) describe for various projects the “Unit Time Value” that represents the cost of delays to the owner. This cost includes the expense of temporary accommodation, loss of profits, the inconvenience to customers, etc.

Margins for construction projects are relatively modest (Potts and Ankrah 2013), and so the net benefit of the project on completion was taken to be 25% of the construction cost, i.e. $5,000,000. Clearly, different types of construction projects will have different benefit percentages. The figure of 25% comes from the latest review (Oakervvee 2020, Berkeley 2020) of the UK’s high speed rail project, HS2, which place it in the low value band of 0% to 50% (DoT 2013). Although this value is much less than the earlier estimate of over 100% (DoT 2013), it is not unusual for large infrastructure projects to undergo similar revisions (Cantarelli et al., 2010).

Table 1 of Herbsman et al. (1995) used a “daily road user cost” (DRUC) of $7,000. As this DRUC does not include indirect costs, the actual disruption cost “will actually
be much higher" (Herbsman et al. 1995, page 274). For example, upgrading a carriageway as in the UK’s smart motorway programme can cause very significant disruption throughout the project. Assuming that the disruption cost approximately equals the construction cost (plus the benefit of the project) when upgrading major highways i.e. the projects in Herbsman et al. (1995), this leads to the DRUC being multiplied by 5 to give $35,000 and a total disruption cost of approximately $25,000,000 for the project. Therefore, the value of the project on completion, $q_0$, was calculated to be the disruption cost plus construction cost plus the ensuing benefit of the project, i.e. $q_0=25,000k+20,000k+5,000k=50,000k$. The sensitivity of the model to $q_0$ is considered in Section 5. While some types of project will not incur the high disruption costs of highways upgrades, there will often be other sources of high disruption costs for significantly late delivery, e.g. for hospitals, power stations, etc.

Hence the weekly $\omega$ was estimated as $\omega = \frac{7 \times 35,000}{50,000,000} \approx 0.005$. The sensitivity of the model to this value is investigated using $\omega = 0.006$ and $\omega = 0.004$ in Section 5. The probability density function, $f_\lambda()$, of the project completion time is modelled as having mean $\mu_\lambda$ and standard deviation $\sigma_\lambda$ where

$$\mu_\lambda = \frac{\mu_1}{\lambda M}$$

with $\mu_1$ being the mean completion time for a resource rate of 1, and $M \in (0, 1)$ models the decrease in efficiency for higher resource rates. For example, doubling the resource rate is unlikely to halve the completion time because the higher resource rate cannot be employed as efficiently as the lower one. In line with the estimated completion time of 700 days from Table 1 in Herbsman et al. (1995) and the use of weeks rather than days, $\mu_1$ was taken to be 100.

$M$ was taken to be 0.8 to model the loss of efficiency if more resources are used at the same time (see Section 2). The sensitivity of the model to this value is investigated using $M = 0.9$ in Section 5.

$$\sigma_\lambda = \sigma \mu_\lambda$$

with $\sigma$, the coefficient of variation, being a constant, i.e. it does not vary with $\lambda$.

Based on Figure 3 of Jackson (2003) and Table 3 of Love et al. (2013), $\sigma$ is taken to be 0.25. Based on Table 1 of Love et al. (2013), the skewness is taken to be 0.4. The sensitivity to the coefficient of variation and the skewness is explored in Section 5.

Therefore
\[ f_\lambda(T) = \lambda^{-M} f_1(T) \]  \hspace{1cm} (4)

Hence there are two elements that the project manager can set and the contractor can agree to, namely \( g \) and \( h \), and a third element, \( \lambda \), that the contractor can choose and whose value will often remain unknown to the project manager.

### 3.1 Distribution used for the analysis

Various distributions have been suggested for modelling project completion times. A key requirement for these distributions would seem to be the ability to model skewness with a short tail on the left and a longer tail on the right (see for example Table 1 in Love et al. 2013). This stems from the fact that the time taken to complete a project is likely to be close to the estimated completion time if things go well, but can be much longer than the estimated time if difficulties are encountered. Hence the normal and exponential distributions that formed the basis of the work reported in Kwon et al. (2010) do not match most situations. Various other distributions have been suggested such as the Erlang (Bendell et al. 1995), Weibull (Abdelkader 2004), lognormal (Trietsch et al. 2012), beta and bicubic (Perez et. 2016). The natural parameters for practitioners to estimate are the earliest and latest possible completion times along with the most likely completion time. When considering PERT analyses, Hajdu and Bokor (2016) found that the results were significantly more sensitive to inaccuracies in the estimates of these three values than the choice between the beta, uniform, triangular and lognormal distributions for modelling the project duration. Therefore, the simplicity of the triangular distribution’s specification along with its straightforward mathematical properties such as its moment generating function, means that it has been used “in numerous papers dealing with the Project Evaluation and Review Technique – PERT” (Kotz and van Dorp (2004), page 5). Consequently, the triangular distribution was chosen for the analysis in this paper. The paper’s triangular distribution notation follows that of Evans et al. (1993) and uses \( a \) for the lowest value, \( b \) for the highest value, and \( c \) for the mode. The properties of the triangular distribution that are used in this paper are given in Appendix A.

For the main analyses in the paper, the triangular distribution’s parameters were set to be \( a=50 \), \( b = 168 \) and \( c = 82 \), giving the base distribution shown in Figure 1. It has mean 100, skewness 0.40 and coefficient of variation 0.25 (see Appendix A). Triangular distributions \( a=70 \), \( b = 141 \) and \( c = 89 \) giving a mean of 100, a skewness of 0.40 and a coefficient of variation of 0.15, and \( a=44 \), \( b = 165 \) and \( c = 92 \) giving a mean of 100, a skewness of 0.20 and a coefficient of variation of 0.25, are also shown in Figure 1. These distributions are used when investigating the sensitivity of the results to the coefficient of variation and the skewness in Section 5.
3.2 Risk aversion

Following Lippman et al. (2013), the utility function of a party who is risk averse is modelled as a negative exponential:

\[ U(x) = 1 - e^{-\psi x} \]  

(5)

where \( \psi \) is the risk aversion factor.

Expanding as a Taylor series gives:

\[ \lim_{\psi \to 0} U(x) = \psi x + o(\psi^2), \]

Hence, for small \( \psi \) the risk averse utility approximately equals the risk neutral utility, i.e. \( U(x) = x \), apart from being multiplied by the constant \( \psi \).

4 Analysis of the optimal contracts

The question of interest is:

How much inefficiency results from using the resource rate \( \lambda \) that stems from the bargaining solution \( g \) and \( h \) values, compared with using a \( \lambda \) value equal to the centrally coordinated value \( \lambda_{\text{co-ord}} \)?

This is important as it is a factor that needs to be taken into account when evaluating the benefits of outsourcing.

From equations 1 and 3, the project manager’s profit, \( W \), is
\[ W = q_0 \left( 1 - \omega T \right) - g + h T \]  
(6)

From equations 1 and 2, the contractor’s profit, \( V \), is

\[ V = g - h T - k \lambda^n T \]  
(7)

As neither the project manager nor the contractor should gain from delaying the “contractual” completion time if the work has been completed, the following constraints are introduced:

\[ h \leq q_0 \omega \] as otherwise, the project manager wants \( T \) as large as possible

and

\[ h \geq 0 \] as otherwise the contractor chooses \( \lambda \approx 0 \) and then wants \( T \) as large as possible

4.1 Resource rate when centrally co-ordinated

Considering the benefit (welfare) to society, equations 2 and 3 give:

\[ \text{benefit} = E\left[q_0 - q_0 \omega T - k \lambda^n T\right] = q_0 - q_0 \omega \mu\lambda - k \lambda^n \mu\lambda \]  
(8)

How this changes as \( \lambda \) changes for the values of Section 3 is shown in Figure 2. Hence the choice of \( \lambda \) can have a significant effect on the percentage of the centrally coordinated benefit that can be achieved.

![Figure 2: How the percentage of the benefit achieved by the project when compared with the maximum possible achievable benefit, varies with the resource rate using the parameters from Section 3.](image-url)
Using $\mu_{\lambda} = \frac{\mu_1}{\lambda^M}$ as in equation 4, equation 8 can be rewritten as

$$\text{benefit} = q_0 - q_0 \omega \frac{\mu_1}{\lambda^M} - k \lambda^{n-M} \mu_1$$

(9)

Differentiating with respect to $\lambda$ and setting the result equal to zero so as to find the optimal value of $\lambda$, gives

$$M q_0 \omega \frac{\mu_1}{\lambda^{M-1}} = (n-M) k \lambda^{n-M-1} \mu_1$$

(10)

and so

$$\lambda = \left( \frac{M q_0 \omega}{(n-M) k} \right)^{\frac{1}{n}}$$

(11)

This is the centrally coordinated resource rate. This equation shows how the optimal value of $\lambda$ varies with the model parameters. Its value is 1.08 for the parameter values of Section 3.

4.2 Risk neutral project manager, risk neutral contractor

The risk neutral project manager’s utility is:

$$U_{PM} = \mathbb{E}[W] = (q_0 - g) + (h - q_0 \omega) \mathbb{E}[T] = (q_0 - g) + (h - q_0 \omega) \mu_{\lambda}$$

(12)

The risk neutral contractor’s utility is:

$$U_{CO} = \mathbb{E}[V] = g - (h + k \lambda^n) \mathbb{E}[T] = g - (h + k \lambda^n) \mu_{\lambda}$$

(13)

Hence, the Nash product is:

$$N = \{(q_0 - g) + (h - q_0 \omega) \mu_{\lambda}\} \times \{g - (h + k \lambda^n) \mu_{\lambda}\}$$

(14)

If either of $U_{PM}$ or $U_{CO}$ is less than zero, then the Nash product is set to zero as a party will not participate if their utility is negative.

If we let

$$r = k \lambda^n$$

(15)

then

$$N = q_0 g - q_0 h \mu_{\lambda} - q_0 r \mu_{\lambda} - g^2 + g h \mu_{\lambda} + g r \mu_{\lambda} + h g \mu_{\lambda} - h^2 \mu_{\lambda}^2 -$$

$$h r \mu_{\lambda}^2 - g q_0 \omega \mu_{\lambda} + h q_0 \omega \mu_{\lambda}^2 + q_0 \omega r \mu_{\lambda}^2$$

(16)

Regarding $\lambda$ as fixed and differentiating with respect to $g$ to find the stationary points gives

$$\frac{\partial N}{\partial g} = q_0 - 2g + 2h \mu_{\lambda} + r \mu_{\lambda} - q_0 \omega \mu_{\lambda}$$

(17)

Equating to zero gives
\[ g = \frac{q_0 + \mu_\lambda - q_0 \omega \mu_\lambda}{2} + h \mu_\lambda \]  

(18)

Fixing the value of \( \lambda \) means that equation 18 can be written as the straight line

\[ g = C_\lambda + h \mu_\lambda \]  

(19)

where \( C_\lambda \) is the constant

\[ C_\lambda = \frac{q_0 + \mu_\lambda - q_0 \omega \mu_\lambda}{2} \]  

(20)

From equation 12 the risk neutral project manager’s utility on this line is:

\[ U_{PM} = q_0 (1 - \omega \mu_\lambda) - C_\lambda \]  

(21)

Similarly, from equation 13 the risk neutral contractor’s utility on this line is:

\[ U_{CO} = C_\lambda + k \lambda^n \mu_\lambda \]  

(22)

Therefore, as \( \lambda \) varies, equation 18 gives a series of parallel lines as shown in Figure 3.

**Figure 3: Examples of the lines defined by equation 18. On each of these lines \( \lambda \), \( U_{PM} \) and \( U_{CO} \) are all constant.**

From equations 21 and 22, the values of \( U_{PM} \) and \( U_{CO} \) are constant along these lines, and so the Nash product is also constant along these lines. This is true irrespective of the project completion time distribution. For example, for Figure 3 \( U_{CO} \) is 528.3 and \( U_{PM} \) is 90.8 on the line corresponding to \( \lambda=0.8 \). The Nash product for line \( \lambda=0.8 \) is 47,900, and for \( \lambda=1.0 \) is 72,000. Hence, the problem is to choose \( \lambda \) to maximise the Nash product value. The constant value of the utilities on these lines will be used to determine the best values of \( h \) in Sections 4.3 and 4.4 covering risk averse and risk neutral pairings.
Expansion of the Nash product in powers of $\lambda$

As the Nash product is constant along the equation 18 lines, the best value of $\lambda$ will be found by considering the Nash value when these lines cross the $g$ axis, i.e. $h=0$. Equation 18 gives the value of $g$ when $h=0$ at the Nash product maximum as

$$g = \frac{q_0 + r \mu_k - q_0 \omega \mu_k}{2}$$

and so substituting for $g$ in equation 16 with $h=0$ gives

$$N = q_0 g - q_0 r \mu_k - g^2 + g r \mu_k - g q_0 \omega \mu_k + q_0 \omega r \mu_k^2$$

and then expanding and simplifying gives

$$4 \, N = \left\{ \mu_1 \left( k \, \lambda^n + q_0 \omega \right) \lambda^{-M} - q_0 \right\}^2$$

The situation of interest is the maximum value of $N$ when

$$q_0 > \mu_1 \left( k \, \lambda^n + q_0 \omega \right) \lambda^{-M}$$

as high values stemming from large $\lambda$ values are not appropriate as they correspond to negative utilities in equations 12 and 13. Hence it is required to find the $\lambda$ that gives the lowest value of

$$\left( k \, \lambda^n + q_0 \omega \right) \lambda^{-M}$$

Differentiating and equating to zero gives:

$$(n - M) \, k \, \lambda^{n-M-1} = M q_0 \omega \lambda^{-M-1}$$

which is the centrally coordinated solution of equation 11.

**Proposition 1**: When the project manager and the contractor are risk neutral, then the optimal choice of $\lambda$ to maximise the Nash product is the same as for the centrally coordinated situation (irrespective of the project completion time distribution).

**Proof**:

This stems from equations 23 to 28.

**End of proof**

**Proposition 2**: When the project manager and the contractor are risk neutral, then their utilities from a Nash bargaining contract of the form of equation 1 will both be equal to $g$ whatever the value of $\lambda$ situation (irrespective of the project completion time distribution).

**Proof**:
h can be set to zero and then g is given by equation 24. Substituting this value of g into equations 12 and 13 leads to

\[ U_{PM} = U_{CO} = \frac{q_0 + r \mu_h - q_0 \omega \mu_h}{2} \]  

(29)

i.e. both utilities are equal to g.

**End of proof**

4.3 **Risk neutral project manager, risk averse contractor**

The project manager’s utility is given by equation 12.

The contractor’s utility is:

\[ U_{CO,A} = \mathbb{E}[1 - e^{-\varphi V}] = 1 - e^{-\varphi g} \mathbb{E}[e^{\varphi h} e^{\varphi r}] \]  

(30)

where

\[ \varphi \]  

is the contractor’s risk aversion level,

and so the Nash product is

\[ N = \left( (q_0 - g) + (h - q_0 \omega) \mu_h \right) \times (1 - e^{-\varphi g} \mathbb{E}[e^{\varphi h} e^{\varphi r}]) \]  

(31)

**Proposition 3**: N is greater at \( h = 0 \) than at any \( h > 0 \) (as all the project duration risk is taken by the project manager). This is true irrespective of the project completion time distribution.

**Proof (descriptive)**:

If this is not the case, then there is a value H which when \( h = H \) leads to the maximum value of N and this value of N is greater than that at \( h = 0 \). If \( h \) moves away from H towards zero along the risk neutral contractor’s line of constant N, i.e. equation 18, then N increases in the risk averse case compared with the risk neutral case as (i) \( h \) instigates a variable penalty on the contractor, (ii) a lower \( h \) reduces the variability and so is better for the risk averse contractor, (iii) a risk neutral contractor is not affected by the reduction in risk from a lower \( h \), and (iv) a risk neutral project manager is not affected by the changing risk in either case. So moving towards \( h = 0 \) on this line, means that the (risk neutral project manager, risk neutral contractor) Nash product stays the same, but the (risk neutral project manager, risk averse contractor) Nash product increases. This increase in N contradicts the choice of H.

**End of proof**

A formal mathematical proof is given in Appendix B. The descriptive proof has parallels in the proof in Chapter 2 of Osborne and Rubinstein (1990) that when splitting a dollar, player 1 gains if player 2 becomes risk averse.
So h=0 and this gives:

\[ \text{N} = \left( (q_0 - g) - q_0 \omega \mu_h \right) \times \left( 1 - e^{-\varphi g} \mathbb{E}[e^{\varphi T}] \right) \tag{32} \]

Equation 32 depends on the probability distribution of the project completion time but only through the moment generating function.

**Numerical analysis using the triangular distribution**

Expanding the moment generating function for the triangular distribution (see Appendix A), gives

\[
\mathbb{E}[e^{\varphi r t}] = \frac{2}{(b-a)(c-a)(b-c)[\varphi]^2} \left\{ -(b-c) e^{\varphi r c} + (b-c)e^{\varphi r a} + (c-a)e^{\varphi r b} - (c-a)e^{\varphi r c} \right\}
\]

which can then be substituted into equation 32.

The direct implementation of equation 33 can have problems when \( \varphi \) tends to zero. Terms such as \( e^{\varphi r a} \) become equal to \( 1 + \epsilon \varphi r a \) where \( \epsilon \varphi r a \) is extremely small and subtracting, for example, \( e^{\varphi r c} \) from \( e^{\varphi r a} \) and then dividing by \( \varphi^2 \) gives poor precision (in Excel). When \( \varphi r \) becomes less than \( 10^{-9} \), then the exponents \( \varphi r a, \varphi r b \) and \( \varphi r c \) are all less than \( 10^{-6} \) (using the values in Section 3).

Therefore, \( e^{\varphi r a}, e^{\varphi r b} \) and \( e^{\varphi r c} \) are approximated by the first three terms of the exponential power series expansion, i.e.

\[
e^x = 1 + x + \frac{x^2}{2!} + o(x^3)
\]

Note that if \( 0 < x < 0.5 \), then

\[
\frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots < \frac{1}{4!} x^4 \left( \frac{1}{1-x} \right) < \frac{2}{4!} x^4.
\]

Therefore, when \( 0 < x < 10^{-6} \),

\[
\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots < \frac{x^3}{3!} + \frac{2 x^4}{4!} < \frac{2 x^3}{3!}
\]

and so, \( 1 - e^x \) is accurately approximated by \( x + \frac{x^2}{2!} \).

On substituting the exponential expansions for \( e^{\varphi r a}, e^{\varphi r b} \) and \( e^{\varphi r c} \) into equation 33, the constant and \( x \) terms cancel out, the \( x^2 \) term cancels down to the constant 1, leaving

\[
\mathbb{E}[e^{\varphi r t}] = 1 + \frac{\varphi r}{(b-a)(c-a)(b-c)} \left\{ b \left( a^2 - c^2 \right) + a \left( c^2 - b^2 \right) + c \left( b^2 - a^2 \right) \right\} + o((\varphi r)^2) \tag{34}
\]
and as \( a, b, c \) and \( r \) are fixed, as \( \varphi \) tends to zero the \((\varphi r)^2\) term can be disregarded. The solid line in Figure 4 shows the loss of benefit as a percentage of the centrally coordinated benefit resulting from the bargaining choice of resource rate for base parameter values of Section 3.

Figure 4 shows that, for the parameters of Section 3, as the contractor becomes more risk averse, then the loss of benefit increases, i.e. the welfare decreases. Reducing the skewness seems to have little effect but reducing the relative spread, i.e. the coefficient of variation, reduces the loss of benefit. This will be considered further in Section 5.

### 4.4 Risk averse project manager, risk neutral contractor

The project manager’s utility is:

\[
U_{PM_A} = \mathbb{E}\left[1 - e^{-\theta W}\right] = 1 - e^{-Bq_0} e^{Bg} \mathbb{E}\left[e^{-\theta hT} e^{Bq_0} \omega^T}\right]
\]

where

\( \theta \) is the project manager’s risk aversion level.

The contractor’s utility is given by equation 13, and so the Nash product is

\[
N = \left(1 - e^{-Bq_0} e^{Bg} \mathbb{E}\left[e^{-\theta hT} e^{Bq_0} \omega^T}\right]\right) \times \left(g - (h + k \lambda^n) \mu_{\lambda}\right)
\]

Proposition 4: The Nash product is maximised when \( h = q_0 \) \( \omega \) as all the project duration risk is taken by the contractor.

Proof (descriptive):
The argument is essentially the same as for proposition 1 with the effect of the change in sign resulting from the risk aversion being for the project manager rather than the contractor, being counteracted by the Nash product being highest when $h$ is at its upper boundary rather than at its lower boundary.

A formal mathematical proof is given in Appendix C.

**End of proof**

Therefore,

$$U_{PM,A} = E\left[1 - e^{\theta(g - q_0)}\right] = \left(1 - e^{\theta(g - q_0)}\right) \tag{37}$$

and the Nash product is

$$N = \left(1 - e^{\theta(g - q_0)}\right) \times \left(g - (q_0 \omega + k \lambda) \frac{\mu_1}{\lambda^A}\right) \tag{38}$$

Unlike equation 32, this does not involve the moment generating function of the project completion distribution, and so we have:

**Corollary 5:** The Nash product is independent of the distribution modelling the completion time.

Differentiating with respect to $\lambda$ so as to find the value of $\lambda$ that maximises the Nash product gives:

$$\frac{\partial N}{\partial \lambda} = \left(1 - e^{\theta(g - q_0)}\right) \times \mu_1 (M q_0 \omega \lambda^{-M-1} - k (n - M) \lambda^n \lambda^{-M-1}) \tag{39}$$

Therefore, $\lambda$ at a maximum satisfies

$$M q_0 \omega \lambda^{-M-1} = k (n - M) \lambda^n \lambda^{-M-1} \tag{40}$$

i.e. equation 10. This gives:

**Corollary 6:** When the project manager is risk averse and the contractor is risk neutral, then the optimal choice of $\lambda$ to maximise the Nash product is the same as for the centrally coordinated situation (irrespective of the project completion time distribution).

Consequently, large contractors relative to the project size, are likely to choose the centrally coordinated resource rate as their large size makes them relatively risk neutral.

### 4.5 Risk averse project manager, risk averse contractor

The project manager’s and the contractor’s utilities are given by equations 30 and 35. Hence the Nash product is:

$$N = \left(1 - e^{-\vartheta q_0} e^{q_0} E\left[e^{\vartheta hT} e^{\vartheta q_0} \omega^T\right]\right) \times \left(1 - e^{-\varphi q_0} E\left[e^{\varphi hT} e^{\varphi q_0} \omega^T\right]\right) \tag{41}$$

Numerical analysis using the triangular distribution
Equation 34 can be used to write this as a function of g, h and λ for the triangular distribution case. The resulting equation is non-linear in g, h and λ.

Figure 5 shows the results of numerically solving for the percentage loss of welfare for different risk aversion levels for the project manager and the contractor. The missing values in the top right of the figure are due to one or both of the utilities being negative for these cases.

![Figure 5: The percentage loss of benefit when both the project manager and the contractor are risk averse.](image)

The special cases of a risk averse project manager & risk neutral contractor, and a risk neutral project manager & a risk averse contractor (Figure 4) respectively correspond to the bottom row and the first column of Figure 5.

Figure 5 indicates that for the parameters of Section 3, if the ratio of the contractor’s risk aversity to the project manager’s risk aversity is low, then the welfare gain corresponds to the centrally coordinated welfare gain. As this ratio increases above 1, less of this centrally coordinated welfare gain is achieved.
5 Sensitivity analysis

Based on the literature, Section 3 chose the values of a number of parameters that were used in the base case analysed in Section 4. This section investigates the generalisability of Section 4’s results by looking at their dependency on the completion time’s distribution in Section 5.1 and the cost parameters in Section 5.2.

5.1 Sensitivity to the completion time’s distribution

5.1.1 Risk neutral project manager – risk averse contractor

Returning to Figure 4, changing the coefficient of variation from 0.25 to 0.15 more than halved the loss of benefit when compared with the coordinated solution. However, changing the skewness value from 0.40 to 0.20 had little effect.

\( U_{PM} \) is given by equation 12 and so, for a specified \( \mu_1 \) and \( \lambda \), it is unaffected by the choice of distribution used to model the completion time. As proposition 3 means that \( h=0 \), for a given distribution equation 30 means that the domain where \( U_{CO,A} \) equals zero is just a function of \( g \) and \( \lambda \). For distributions with analytic moment generating functions, these “\( U_{CO,A} \) equals zero” functions of \( g \) and \( \lambda \) can easily be compared. A significant difference between these functions would indicate that the choice of distribution has an effect on the analysis. The triangular distribution was compared with the gamma distribution for the base case values of Section 3 and with the distributions having the same mean and standard deviation. The gamma distribution was chosen for this comparison as it has an unbounded right tail. There was very little difference between the two “\( U_{CO,A} \) equals zero” functions for \( \varphi \) up to 0.03. However, for larger values of \( \varphi \) the risk caused by the gamma distribution’s unboundedness, meant that there could be no region where both \( U_{PM} \) and \( U_{CO,A} \) were positive. For example, Figure 6 shows the case for \( \varphi=0.05 \) (and \( \omega=0.008 \), and \( q_0 \) being twice the value of Section 3). \( U_{CO,A} \) is positive above the \( U_{CO,A} \) zero curve and \( U_{PM} \) is positive below the \( U_{PM} \) zero curve. These regions intersect for the triangular distribution but not for the gamma distribution. Consequently, using unbounded distributions for the analysis can lead to problems.
Figure 6: $U_{PM}$ is positive below the $U_{PM}$ zero curve while $U_{CO_A}$ is positive above the $U_{CO_A}$ zero curve. Hence, only the triangular distribution has a region where both utilities are positive. ($\phi=0.05$, $\omega=0.008$)

5.1.2 Risk averse project manager – risk averse contractor

Figure 7 The percentage loss of benefit compared with the centrally coordinated case when the coefficient of variation is reduced to 0.15.
Figures 7 and 8 consider the coefficient of variation value of 0.15 and skewness value of 0.20 for the situation when both the project manager and the contractor are risk averse. Comparing these figures with Figure 5, then as found in Section 5.1.1, the results are sensitive to the coefficient of variation but relatively insensitive to the skewness. It is also noticeable that the Nash product is positive over the whole of Figure 7 unlike Figures 5 and 8 where it goes negative in the top right hand corner.

5.2 Sensitivity to the cost parameters

< SINGLE COLUMN WIDTH> Figure 8 The percentage loss of benefit compared with the centrally coordinated case when the skewness is reduced to 0.20.
Figures 5, 9 and 10 all use the “Base” triangular distribution. The only difference is instead of using $n=2$ and $A=0.8$ as in Figure 5, Figure 9 uses $n=1.4$ and Figure 10 uses $A=0.9$. Comparing Figures 9 and 10 with Figure 5 shows that changing $M$ from 0.8 to 0.9 has little effect, but going from $n=2$ to $n=1.4$ increases the loss of benefit compared with the coordinated situation. It is noticeable that the loss of benefit is occurring when $\phi$ becomes a bit larger than $\theta$ whereas in Figure 5 this loss is when $\phi$ becomes much larger than $\theta$. 
Figure 10 The percentage loss of benefit when $M$, the power of the resource rate when defining $\mu_\lambda$ in equation 4, is increased to 0.9.
Figure 11 The percentage loss of benefit when $\omega$, the proportion of the project value lost with a delay (equation 3), is reduced to 0.004.

Figures 5, 11 and 12 consider the "Base" triangular distribution with $\omega$ values of 0.005, 0.004 and 0.006 respectively. They show that the loss of benefit is very sensitive to the relative sizes of $\theta$, $\varphi$ and $\omega$. When $\omega$ is 0.006, not only is the loss of benefit high when $\theta$ is small, but the Nash product is negative for many combinations of $\theta$ and $\varphi$. While the loss of benefit for $\omega=0.004$ (Figure 11) is lower than when $\omega=0.005$, it shows the same pattern and mirrors the bottom left hand quadrant of Figure 5.
Figure 12 The percentage loss of benefit when $\omega$, the proportion of the project value lost with a delay (equation 3), is increased to 0.006.
Figure 13 The percentage loss of benefit when $k$ in equation 5 which defines the contractor’s cost as $k\lambda^n$, is increased from Figure 5’s value of 2.0 to 2.5.

Figure 13 illustrates how increasing $k$ from 2.0 to 2.5 in equation 2, i.e. the contractor's cost = $k\lambda^n$, leads to a loss of benefit compared with Figure 5. Additionally, the region where there is no solution as the Nash product is negative, has significantly increased. Reducing $k$ to 1.5 leads to the grid being virtually all black with just a handful of purple cells in the top left corner.
Comparing Figures 5 and 14, shows that increasing the project benefit by doubling $q_0$ reduces the percentage loss of benefit when the contractor is considerably more risk averse than the project manager, but otherwise the change has a limited effect. However, reducing $q_0$ to 75% of its base value in Section 3, led to the grid becoming all black, i.e. the percentage loss of benefit was negligible.

6 Modelling highways construction contracts

The model considered in Sections 3, 4 and 5 investigated the linear time based contract where the constant and linear parameters are determined by the Nash bargaining solution. Herbsman et al. (1995) describe four innovative contracting methods being introduced into highways construction contracts in the United States. The approaches of bidding a cost and a time with no penalties for late completion, and lane rental fall outside the scope of this paper. However, the approaches of having an incentive for early completion and a disincentive for late completion, and bidding a cost and a time and then having incentives and disincentives for early and late delivery
are closely related to the models considered in Sections 3 to 5. Hence, these models are considered in Sections 6.1 and 6.2 respectively.

6.1 Incentive / disincentive contracts

The project manager decides on the daily cost with the payment to the contractor decreasing by this amount for every day over the target completion time and increasing by this amount for every day under it. Hence, h for the model of equation 1 is fixed at \( q_0 \omega \) in equation 3, the “daily road user cost” (Herbsman et al. 1995, page 274). Consequently, all the risk is passed on to the contractor (see also Proposition 4).

The financial outcome for each party is:

Project Manager: \( q_0 - g \)  \( (42) \)

Contractor: \( g - q_0 \omega T - k \lambda^n T \) where \( \mathbb{E}[T] = \mu_{\lambda} = \frac{\mu_1}{\lambda M} \)  \( (43) \)

6.1.1 Risk neutral contractor

Differentiating equation 14 and taking \( h=q_0 \omega \) to find the optimal value of \( g \) leads to equation 18 becoming

\[
g = \frac{1}{2} \left( q_0 + k \lambda^n \mu_{\lambda} + q_0 \omega \mu_{\lambda} \right) \tag{44}\]

and then equation 14 gives

\[
N = \frac{1}{4} \left( q_0 - k \lambda^n \mu_{\lambda} - q_0 \omega \mu_{\lambda} \right)^2 \tag{45}\]

As this is the same as equation 25, the approach of equations 26 to 28 applies and the value of \( \lambda \) that maximises equation 45 is the value given in equation 11. Hence, it is the same as the centrally coordinated solution.

6.1.2 Risk averse contractor

After substituting \( h=q_0 \omega \) into the Nash product equation 31, the value of \( g \) is the value that maximises:

\[
N = (q_0 - g) \times \left( 1 - e^{-\varphi g} \mathbb{E} \left[ e^{\varphi (q_0 \omega + k \lambda^n)T} \right] \right) \tag{46}\]

For any value of \( g \), the value of \( \lambda \) that maximises equation 46 is given by differentiating \( \mathbb{E} \left[ e^{\varphi (q_0 \omega + k \lambda^n)T} \right] \) with respect to \( \lambda \) and then setting the result to zero.

\[
\frac{\partial \mathbb{E} \left[ e^{\varphi (q_0 \omega + k \lambda^n)T} \right]}{\partial \lambda} = \frac{\partial}{\partial \lambda} \int_{0}^{\infty} f_{\lambda}(T) \ e^{\varphi (q_0 \omega + k \lambda^n)T} \ dT \tag{47}\]
From equation 4, \( f_\lambda(T) = \lambda^{-M}f_1(T) \) and so

\[
\frac{\partial \mathbb{E}[\exp(q_0 \omega + k\lambda T)]}{\partial \lambda} = \frac{\partial}{\partial \lambda} \int_0^\infty \lambda^{-M}f_1(T) \ e^{q_0 \omega + k\lambda T} \ dT = -M \lambda^{-(M+1)} \int_0^\infty f_1(T) \ e^{q_0 \omega + k\lambda T} \ dT + \lambda^{-M} n \ k \lambda^n \int_0^\infty f_1(T) \ e^{q_0 \omega + k\lambda T} \ dT = 0
\]

(48)

This leads to equation 11: \( \lambda = \left( \frac{M q_0 \omega}{(n-M) k} \right)^{\frac{1}{n}} \), i.e. the resource rate is the same as that for the centrally coordinated case.

### 6.2 Bidding on cost / time combined with incentive / disincentive

As in Section 6.1, \( h \) is fixed at \( q_0 \omega \), and again all the risk is passed on to the contractor. As the contractor’s bid values are \( A \) and \( B \), the payment between the parties is:

\[
A + (B - T) q_0 \omega
\]

(49)

and the financial outcome for each party is:

- **Project Manager**: \( q_0 - A - B q_0 \omega \)
  
  (50)

- **Contractor**: \( A + B q_0 \omega - q_0 \omega T - k \lambda^n T \) where \( \mathbb{E}[T] = \mu_\lambda = \frac{\mu_1}{\lambda^M} \)

(51)

So in equation 6, \( g \) and \( h \) are:

\[
g = A + B q_0 \omega
\]

(52)

\[
h = q_0 \omega
\]

(53)

#### 6.2.1 Risk neutral contractor

Equation 14 becomes

\[
N = (q_0 - A - B q_0 \omega) \times \{A + B q_0 \omega - (q_0 \omega + k \lambda^n) \mu_\lambda\}
\]

(54)

Differentiating with respect to \( \lambda \) gives:

\[
\frac{dN}{d\lambda} = -\left(q_0 - A - B q_0 \omega\right) \mu_1 \left(\frac{-M q_0 \omega}{\lambda^{M+1}} + (n - M) k \lambda^{n-M-1}\right)
\]

(55)

Setting equation 53 to zero gives

\[
M q_0 \omega = (n - M) k \lambda^n
\]

(56)

which is equation 11, and so \( \lambda \) equals the centrally coordinated value.
6.2.2 Risk averse contractor

Equation 31 becomes

\[ N = (q_0 - A - B q_0 \omega) \times \left( 1 - e^{-\varphi A} e^{-\varphi B q_0 \omega} \mathbb{E}[e^{\varphi(q_0 \omega + r)T}] \right) \]  

(57)

The only variable is \( \lambda \) and so the situation is the same as equation 46, and again the optimal \( \lambda \) is given by equation 11:

\[ \lambda = \left( \frac{M q_0 \omega}{(n-M) k} \right)^{1/2} \]

6.3 Comments

Choosing the time penalty to be \( h = q_0 \omega \) leads to \( \lambda \) being chosen to have the same value as the centrally coordinated resource rate irrespective of the risk preferences of the project manager and the contractor. However, for the case of a risk averse contractor, this choice of \( h \) leads to a lower Nash product as shown by proposition 3, as \( h = q_0 \omega \) means that the contractor ends up accepting all the risk. Figure 15 shows the contractor’s utility, \( U_{co} \), for the contracts of Section 6.2 divided by \( U_{co} \) from the time based contracts examined in this paper (i.e. Section 4.3). As the contractor’s risk aversion increases, the Section 6.2 contracts become less and less attractive compared to the contractor compared with those of Section 4.3. For the project manager, this \( U_{pm} \) ratio decreases as the contractor’s risk aversion increases but levels off at around 90% for all three triangular distributions.

![Figure 15](image)

< SINGLE COLUMN WIDTH> Figure 15 How the ratio of the \( U_{co} \) from Section 6.2 divided by the \( U_{co} \) from Section 4.3 decreases as the contractor’s risk aversion increases for the base and coefficient of variation equal to 0.15 triangular distributions. The skewness equal to 0.2 triangular distribution results were very similar to the base results, and so they are not shown.
7 Discussion and conclusions

Having a completion time incentive in project contracts is a natural way to reduce the chances of a project being completed late (Herbsman et al. 1995, Arditi et al. 1997, Meng & Gallagher, 2012, Chen & Lee 2017). This paper has considered contracts where both the fixed payments and the time incentive payments are chosen to maximise the Nash product of the project manager’s and the contractor’s utilities. Hence, the contracts are more general than those where the time incentive payments are fixed before the negotiation as in Herbsman et al. (1995), Arditi et al. (1997) and Shr & Chen (2003).

A main finding of the analysis is that:

- The Nash bargaining solution coordinates the resources when the contractor is risk neutral whether or not the project manager is risk neutral or risk averse.

This result is true for all project completion time distributions (Proposition 1 and Corollary 6).

If the contractor is less risk averse than the project manager, then the regions below the leading diagonal in Figures 5, and 7 to 12 show that there seems to be little or no loss of benefit compared to using the centrally coordinated resource rate. If the contractor is considerably more risk averse than the project manager, then a significant loss of benefit occurs as shown by the upper left section of Figures 5, and 7 to 12. Comparing Figure 5 with skewness 0.40 to Figure 8 with skewness 0.2 suggests that the loss of benefit is not that sensitive to the skewness level. Similarly, Figures 5, 9 and 10 suggest that the sensitivity to the parameters n and M is low. However, comparing Figures 5 and 7 shows that higher coefficients of variation lead to greater losses, and Figures 5, 11 and 12 show that the loss of benefit is very sensitive to $\omega$, the proportion of the project value lost for every day the project takes. However, these results should be regarded as indicative rather than definitive as they are based on the triangular distribution and a limited number of parameter values.

Consequently, the relative risk aversity of the contractor compared to the project manager is an important factor to consider when outsourcing projects. Where the contractor is more risk averse than the project manager, then the contracts of Section 4 lead to a choice of resource rate that gives a loss of efficiency compared with the centrally coordinated resource rate. On the other hand, the contracts of Section 6 choose the resource rate to be the centrally coordinated resource rate but the resulting utilities are lower, and so less attractive to the participants. A situation where this is likely to occur is with government contracts where the project manager, i.e. the government, is likely to be less risk averse than the contractor.

Hence, other main findings of the analysis are that:
• If the contractor is more risk averse than the project manager, then a significant loss of benefit can occur.

• If the contractor is less risk averse than the project manager, then little loss of benefit seems to occur.

Section 6 considered contracts where the incentive rate is fixed and only the fixed term in contract payment is negotiable. Herbsman et al. (1995) detail the use of these contracts for constructing and refurbishing highways in the USA. It was found that although these contracts led to the chosen resource rate being the same as the centrally coordinated resource rate, they gave lower individual utilities, and so a lower Nash product than the contracts analysed in Section 4. Figure 15 shows that the contractor’s utility for Section 6 contracts, decreases rapidly compared to Section 4 contracts as the contractor’s risk aversion increases. Consequently, Section 4’s contracts would seem much more appropriate in the situation where the contractor is more risk averse than the project manager, i.e. Section 6’s contracts favour large contractors over smaller ones.

Besides these findings, this paper has extended academic knowledge by:

• Analysing the Nash bargaining approach for time based project contracts.

• Providing a novel general proof of Proposition 3 that the optimal solution is for \( h \) to be zero in the case of a risk neutral project manager and risk averse contractor that applies to all distributions. This compares with Kwon et al. (2010) where the proof was restricted to the normal and exponential distributions.

• Considering risk averse project managers.

There is growing awareness that besides profit maximisation, fairness considerations also play a part in supply chain negotiations (Rabin 1993; Choi & Messinger 2016; Meng et al. 2018; Sarkar 2019). This can be accentuated when considering large projects, such as constructing a new hospital, as often only a small number of contractors have the expertise and resources to undertake the project. Hence maintaining good working relationships with the potential contractors is important. Therefore, future work extending the analysis to incorporate fairness into the model would be valuable and timely.

References


**Appendix A: Properties of the triangular distribution**

The triangular distribution with parameters of a, b and c (the bottom, top and mode point respectively) has the following properties:

- The mean is \( \mu = \frac{a+b+c}{3} \)

- The variance is \( \sigma^2 = \frac{a^2 + b^2 + c^2 - a b - a c - b c}{18} \)

- The skewness is \( \gamma = \frac{\sqrt{2} (a + b - 2c) (2a - b - c) (a - 2b + c)}{5 \left( a^2 + b^2 + c^2 - a b - a c - b c \right)^{3/2}} \)
the probability density function is \( f(x) = \frac{2}{(b-a)(c-a)} (x-a) \) for \( x \in [a, c] \)

the moment generating function is

\[
\mathbb{E}[e^s] = \frac{2}{(b-a)(c-a)} \left[ \int_{t=0}^{c} \left( \frac{t-a}{s} e^{s t} \right) dt + \int_{t=c}^{b} \left( \frac{b-t}{s} e^{s t} \right) dt \right] = \frac{2}{(b-a)} \left[ \left( \frac{1}{(c-a)} \left( \frac{t-a}{s} - \frac{t^2}{2s^2} \right) \right) \bigg|_{t=c}^{c} + \left( \frac{1}{(b-c)} \left( \frac{b-t}{s} + \frac{t^2}{2s^2} \right) \right) \bigg|_{t=b}^{c} \right]
\]

Appendix B:

Proof of proposition 3 (risk neutral project manager, risk averse contractor)

For any \( \lambda \), the claim is that \( N \) strictly decreases as \( h \) increases on the interval \([0, q_0 \omega]\).

Suppose this is not the case, then there exists a value \( H \) such that \( N(h) \leq N(H) \) for all \( h \in [H - \delta, H] \) for some \( \delta > 0 \). Define \( H^* \) by \( H^* = H - \delta \). The value of \( g \) at \( H \) given by equation 18 will be denoted by \( G \), while that of \( g \) at \( H^* \) will be denoted by \( G^* \).

The utility for a risk averse contractor is (from equation 30):

\[
U_{CO_A} = 1 - e^{-\varphi q} \mathbb{E}[e^{\varphi h^T \varphi r^T}] \tag{B.1}
\]

where

\( \mu_\lambda \) has been shortened to \( \mu \) as \( \lambda \) is fixed

\( q \) has been used to denote \( q_0 - q_0 \omega \mu_\lambda \).

\((G^*, H^*)\) was chosen so that it lies on the line for which \( U_{PM,N} \) and \( U_{CO,N} \) do not change (see equation 18). As the assumption that

\[
N(G^*, H^*) \leq N(G, H) \tag{B.2}
\]

is disproved if

\[
N(G^*, H^*) > N(G, H) \tag{B.3}
\]

this is equivalent to

\[
U_{CO_A}(G^*, H^*) > U_{CO_A}(G, H) \tag{B.4}
\]

Substituting equation B.1 into equation B.4 gives

\[
1 - e^{-\varphi G} \mathbb{E}[e^{\varphi H^T \varphi r^T}] > 1 - e^{-\varphi G} \mathbb{E}[e^{\varphi H^T \varphi r^T}] \tag{B.5}
\]
Equation B.5 can be rewritten as
\[
e^{-\varphi G} E[e^{\varphi HT} e^{\varphi T}] > e^{-\varphi G^*} E[e^{\varphi (H-\delta \mu)T} e^{\varphi T}]
\]  \hspace{1cm} (B.6)

Writing \( G^* = G - \varepsilon \), then equation 19 means that \( \varepsilon = \delta \mu \). Substituting for \( G^* = G - \delta \mu \) and \( H^* = H - \bar{H} \) gives
\[
e^{-\varphi G} E[e^{\varphi HT} e^{\varphi T}] > e^{-\varphi (G-\delta \mu)} E[e^{\varphi (H-\delta \bar{H})T} e^{\varphi T}]
\]  \hspace{1cm} (B.7)

Hence
\[
e^{-\varphi \delta \mu} E[e^{\varphi HT} e^{\varphi T}] - E[e^{\varphi (H-\delta \bar{H})T} e^{\varphi T}] > 0
\]  \hspace{1cm} (B.8)

As \( \delta \) tends to zero:
\[
\lim_{\delta \to 0} e^{-\varphi \delta \mu} = 1 - \delta \varphi \mu + O(\delta^2)
\]  \hspace{1cm} (B.9)

Consequently,
\[
\lim_{\delta \to 0} \left\{ e^{-\varphi \delta \mu} E[e^{\varphi HT} e^{\varphi T}] - E[e^{-\varphi \delta T} e^{\varphi (H+\delta T)}] \right\} =
\lim_{\delta \to 0} \left\{ -\delta \varphi \mu E[e^{\varphi (H+\delta T)}] - E[-\varphi \delta T e^{\varphi (H+\delta T)}] + O(\delta^2) \right\} =
\lim_{\delta \to 0} \left\{ -\delta \varphi (\mu E[e^{\varphi (H+\delta T)}] - E[T e^{\varphi (H+\delta T)}]) + O(\delta^2) \right\}
\]  \hspace{1cm} (B.10)

Hence as \( \delta \) tends to zero, equation B.8 gives
\[
-\mu E[e^{\varphi (H+\delta T)}] - E[-T e^{\varphi (H+\delta T)}] > 0
\]  \hspace{1cm} (B.11)

Let
\[
Z = \varphi (H + r)
\]  \hspace{1cm} (B.12)

As \( \varphi \), \( H \) and \( r \) are all positive, then so is \( Z \). Equation B.11 can now be rewritten as
\[
E \left[ (T - \mu) e^{Z T} \right] > 0
\]  \hspace{1cm} (B.13)

If we denote the probability density function of \( T \) by \( f() \), then the left hand side of equation (B.13) becomes
\[
\int_0^\infty f(T) \left\{ (T - \mu) e^{Z T} \right\} dT = \int_0^\mu f(T) \left\{ (T - \mu) e^{Z T} \right\} dT + \int_\mu^\infty f(T) \left\{ (T - \mu) e^{Z T} \right\} dT
\]  \hspace{1cm} (B.14)

However, as
\[
\int_0^\mu f(T) \left\{ (T - \mu) e^{Z T} \right\} dT > \int_0^\mu f(T) \left\{ (T - \mu) e^{Z \mu} \right\} dT
\]  \hspace{1cm} (B.15)

because \( (T - \mu) \) is negative and \( e^{Z \mu} \) is bigger than \( e^{Z T} \), and
\[
\int_{\mu}^{\infty} f(T) \left\{ (T - \mu) e^{Z T} \right\} dT > \int_{\mu}^{\infty} f(T) \left\{ (T - \mu) e^{Z \mu} \right\} dT \tag{B.16}
\]
because \((T - \mu)\) is positive and \(e^{Z \mu}\) is less than \(e^{Z T}\), and as
\[
\int_{0}^{\mu} f(T) \left\{ (T - \mu) e^{Z \mu} \right\} dT + \int_{\mu}^{\infty} f(T) \left\{ (T - \mu) e^{Z \mu} \right\} dT = e^{Z \mu} \int_{0}^{\mu} f(T) (T - \mu) dT = 0 \tag{B.17}
\]
from the definition of \(\mu\), then equation B.14 is positive and so equation B.13 is true. Consequently, \(H^*\) is a value of \(h\) near but less than \(H\) that gives a higher Nash product. This contradicts the claim about \(H\), and so the proposition is true.

**End of proof of proposition 3**

**Appendix C:**

**Proof of proposition 4 (risk averse project manager, risk neutral contractor)**

For any \(\lambda\), the claim is that \(N\) strictly increases as \(h\) increases on the interval \([0, q_0 \omega]\).

Suppose this is not the case, then there exists a value \(H\) such that \(N(h) \geq N(h)\) for all \(h \in (H, H + \delta]\) for some \(\delta > 0\). Define \(H^*\) by \(H^* = H + \delta\). The value of \(g\) at \(H\) given by equation 18 will be denoted by \(G\), while that of \(g\) at \(H^*\) will be denoted by \(G^*\).

The utility for a risk averse project manager is (from equation 35):

\[
U_{PM_A} = \mathbb{E} \left[ 1 - e^{-\theta WT} \right] = 1 - e^{-\theta q_0} e^{\theta q_0} \mathbb{E} \left[ e^{-\theta g h T} e^{\theta q_0 \omega T} \right] \tag{C.1}
\]

where

\(\mu_{\lambda}\) has been shortened to \(\mu\) as \(\lambda\) is fixed

\(q\) has been used to denote \(q_0 - q_0 \omega \mu_{\lambda}\).

\((G^*, H^*)\) was chosen so that it lies on the line for which \(U_{PM_N}\) and \(U_{CO_N}\) do not change (see equation 18). As the assumption that

\[
N(G^*, H^*) \leq N(G, H) \tag{C.2}
\]
is disproved if

\[
N(G^*, H^*) > N(G, H) \tag{C.3}
\]
this is equivalent to

\[
U_{PM_A}(G^*, H^*) > U_{PM_A}(G, H) \tag{C.4}
\]
Substituting equation C.1 into equation C.4 gives
\[ 1 - e^{-\theta q_0} e^{B_1} E\left[e^{-\theta H^T} e^{B_q 0 \omega T}\right] > 1 - e^{-\theta q_0} e^{B_1} E\left[e^{-\theta H^T} e^{B_q 0 \omega T}\right] \quad (C.5) \]

Equation C.5 can be rewritten as
\[ e^{BG} E\left[e^{-\theta H^T} e^{B_q 0 \omega T}\right] > e^{BG} E\left[e^{-\theta H^T} e^{B_q 0 \omega T}\right] \quad (C.6) \]

Writing \( G^* = G + \epsilon \), then equation 19 means that \( \epsilon = \delta \mu \). Substituting for \( G^* = G + \delta \mu \) and \( H^* = H + \delta \) gives
\[ e^{BG} E\left[e^{-\theta H^T} e^{B_q 0 \omega T}\right] > e^{G^*} E\left[e^{-\theta H^T} e^{B_q 0 \omega T}\right] \quad (C.7) \]

Hence
\[ e^{-\theta \delta \mu} E\left[e^{-\theta H^T} e^{B_q 0 \omega T}\right] - E\left[e^{-\theta (H + \delta)T} e^{B_q 0 \omega T}\right] > 0 \quad (C.8) \]

As \( \delta \) tends to zero:
\[ \lim_{\delta \to 0} e^{-\theta \delta \mu} = 1 - \theta \mu + O(\delta^2) \quad (C.9) \]

Consequently,
\[ \lim_{\delta \to 0} \left\{ e^{-\theta \delta \mu} E\left[e^{-\theta H^T} e^{B_q 0 \omega T}\right] - E\left[e^{-\theta \delta T} e^{-\theta H^T} e^{B_q 0 \omega T}\right] \right\} = \]
\[ \lim_{\delta \to 0} \left\{ - \theta \mu E\left[e^{\theta(-H + q_0 \omega)T}\right] - E\left[-\theta \delta T e^{\theta(-H + q_0 \omega)T}\right] + O(\delta^2) \right\} \]
\[ \lim_{\delta \to 0} \left\{ - \theta \mu E\left[e^{\theta(-H + q_0 \omega)T}\right] - E\left[T e^{\theta(-H + q_0 \omega)T}\right] \right\} + O(\delta^2) \quad (C.10) \]

Hence as \( \delta \) tends to zero, equation C.8 gives
\[ -\mu E\left[e^{\theta(-H + q_0 \omega)T}\right] - E\left[-T e^{\theta(-H + q_0 \omega)T}\right] > 0 \quad (C.11) \]

Let
\[ J = \theta (-H + q_0 \omega) \quad (C.12) \]

As \( \theta \) and \( (-H + q_0 \omega) \) are both positive, so is \( J \). Equation C.11 can now be rewritten as
\[ E\left[(T - \mu) e^J T\right] > 0 \quad (C.13) \]

If we denote the probability density function of \( T \) by \( f() \), then the left hand side of equation (C.16) becomes
\[ \int_0^\infty f(T) \left\{ (T - \mu) e^J T \right\} dT = \int_0^\mu f(T) \left\{ (T - \mu) e^J T \right\} dT + \int_\mu^\infty f(T) \left\{ (T - \mu) e^J T \right\} dT \quad (C.14) \]

However, as
\[ \int_{0}^{\mu} f(T) \left\{ (T - \mu) \ e^{J \ T} \right\} dT > \int_{0}^{\mu} f(T) \left\{ (T - \mu) \ e^{J \ \mu} \right\} dT \]  
(C.15)

because \((T - \mu)\) is negative and \(e^{J \ \mu}\) is bigger than \(e^{J \ T}\), and

\[ \int_{\mu}^{\infty} f(T) \left\{ (T - \mu) \ e^{J \ T} \right\} dT > \int_{\mu}^{\infty} f(T) \left\{ (T - \mu) \ e^{J \ \mu} \right\} dT \]  
(C.16)

because \((T - \mu)\) is positive and \(e^{J \ \mu}\) is less than \(e^{J \ T}\), and

\[ \int_{0}^{\mu} f(T) \left\{ (T - \mu) \ e^{J \ \mu} \right\} dT + \int_{\mu}^{\infty} f(T) \left\{ (T - \mu) \ e^{J \ \mu} \right\} dT = e^{J \ \mu} \int_{0}^{\infty} f(T) (T - \mu) \ dT = 0 \]  
(C.17)

from the definition of \(\mu\), then equation C.14 is positive and so equation C.13 is true. Consequently, \(H^*\) is a value of \(h\) near but more than \(H\) that gives a higher Nash product. This contradicts the claim about \(H\), and so the proposition is true.

**End of proof of proposition 4**