Spatially Driven Chemical Species Tomography with Size-Adaptive Hybrid Meshing Scheme

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Abstract—This paper develops a size-adaptive hybrid meshing scheme for Chemical Species Tomography (CST) that is driven by the customized spatial resolution of the sensing region. Traditionally, the entire sensing region in CST is uniformly discretized with the empirically determined density of the meshes. Such a discretization results in a) waste of computational efforts on the less spatially resolved location; and b) much severer rank deficiency. To solve the above-mentioned issues, we introduce, for the first time, a size-adaptive hybrid meshing scheme for CST. Driven by the spatial resolution, dense meshes are deployed in the region of interest (RoI) to detail the target flow field while sparse ones are deployed out of the RoI to fully consider the physically existing laser absorption. The proposed scheme is numerically validated using a CST sensor with 128 laser beams. The visual and quantitative metric comparisons show that the proposed hybrid-size meshing scheme outperforms the traditionally uniform-size meshing scheme, giving 35% lower image error and 38% less significant dislocation at a typical 35 dB signal-to-noise ratio in the RoI. The proposed hybrid-size meshing scheme significantly facilitates the reconstruction of the industrial combustion processes where the combustion zone is bypassed by cooling flows. In these scenarios, the proposed scheme can adapt a finer resolution to detail the combustion zone, while maintaining the integrity of the physical model by less resolved reconstruction of the bypass flows.

Index Terms— Chemical Species Tomography, tunable diode laser absorption spectroscopy, region of interest, mesh, spatial resolution.

I. INTRODUCTION

Chemical Species Tomography (CST) has become one of the most representative techniques used in combustion diagnosis for rapid imaging of unknown two-dimensional (2D) distributions of flow parameters, such as species concentration and temperature. CST is implemented by multiple line-of-sight tunable diode laser absorption spectroscopy (TDLAS) measurements, in a manner analogous to x-ray tomography. Its robustness and minimal intrusiveness enable CST a highly desired and in situ solution for industrial applications, e.g., vapor fuel imaging in internal combustion engines [1, 2], aero-engine lean blowout diagnosis [3], and gas turbine exhaust imaging [4, 5].

The Region of Interest (RoI) in CST is regarded as the area where distributions of the target parameters are to be retrieved. The choice of RoI is typically implemented in two ways: (a) covering the whole sensing region with laser beams evenly arranged in the sensing region [6, 7] or (b) covering only the location-predetermined target flow with laser beams densely arranged in this area [8-10].

For case (a), the limited optical access generally restricts the number of beams that can be deployed in industrial practice, making the CST inverse problem inherently rank deficient. With uniform-size meshes, beam layout with limited projections worsens the ill-posedness of the CST inverse problem, resulting in significant spike noises and artefacts in image reconstruction.

For case (b), previous attempts used nitrogen to purify the out-of-RoI area [8] or assumed the small absorption out of the RoI can be neglected [9]. However, the former is less practical to be deployed on industrial combustors, while the latter suffers from reconstruction errors due to the physical existence of out-of-RoI heat dissipation and species convection.

One approach to mitigate the problems mentioned above is to apply a non-uniform meshing scheme for alleviation of rank deficiency, which has been validated in Electrical Capacitance Tomography (ECT) and Electrical Impedance Tomography (EIT). For instance, Wang et al. [11] deployed the adaptive refinement with triangular-shaped grids in ECT, improving the spatial resolution efficiency compared with the uniform refinement and showing more accurate boundary in the reconstruction. Similarly, Molinari et al. [12] developed a self-adaptive refinement algorithm in EIT for simple head model reconstruction based on the error estimation. As ECT and EIT are soft-field tomographic techniques, they are naturally different from the hard-field CST, and therefore, the adaptive refinement mentioned above cannot be directly applied to CST. In addition, most of the target flow fields, for
example, combustion processes, in CST are diffused with their patterns changing rapidly. This characteristic is radically different from that in ECT and EIT where clear boundaries exist for the inhomogeneity. Consequently, the non-uniform meshing schemes proposed in ECT and EIT are inadequate to characterize the naturally diffused flows in CST.

Although a couple of previous works reported CST with irregular sensing regions, these attempts still employed a uniform-size or nearly uniform-size meshing without customization of the density of the grids towards the target flows. For example, Wood et al. [13] numerically demonstrated the reconstruction of an annular sensing region for turbofan engines imaging, but with uniform-size meshes in the domain. Recently, Grauer et al. [14] applied the finite element in meshing with Bayesian model for image reconstruction in irregular region. Although the finite element meshing scheme enables better flexibility at the domain boundary, the mesh sizes in the sensing domain are still nearly uniform size without specific or customized discretization of the target area.

In this paper, we introduce a size-adaptive hybrid meshing scheme for CST image reconstruction. The new scheme has four main contributions:

(a) It introduces a spatially driven strategy to determine quantitatively the mesh size.
(b) It is computation-effective by prioritizing detailed reconstruction of the target flow.
(c) It maintains the physical integrity by fully considering the background absorption into reconstruction.
(d) It mitigates the impact of noise by self-adaptive refactorization of the sensing matrix.

This development significantly facilitates the industrial application of CST towards practical combustors without modifying the layout of the optical sensor. In case of imaging combustion zone that is bypassed by cooling flows, e.g. aero-engine exhaust imaging [15], the proposed scheme can better characterize the target combustion zone with dense meshes, while maintaining the integrity of the physical model by considering the absorption in the bypass flows with sparse meshes.

The rest of the paper is structured as follows: The proposed scheme is mathematically analyzed in Section II. Numerical simulation and experiments are carried out to validate the proposed meshing scheme in Section III and Section IV, respectively. The paper is concluded in Section V.

II. METHODS

A. Mathematical Formulation of CST

The fundamentals of CST and its rank deficiency are firstly reviewed to facilitate the introduction of the size-adaptive hybrid meshing scheme we propose in the next subsection. As shown in Fig. 1, the laser sensing area that covers the entire optical paths from emitters to detectors is named as Region of Sensing (RoS) here afterwards.

Given the RoS is discretized into $N$ pixels, temperature, pressure, and mole fraction (concentration) of the absorbing species are assumed to be uniform in each pixel. According to Beer’s law, the integrated absorbance $b_i$ of the $i$-th laser beam for the target species is given by

$$b_i = \sum_{j=1}^{N} A_{ij}k_j,$$

where $A_{ij}$ represents the chord length of the $i$-th beam within the $j$-th pixel, $k_j$ the absorption density of the $j$-th pixel at the selected absorption transition [16]. $k_j$ is defined by

$$k_j = P_j x_j S(T_j),$$

where $P_j [\text{atm}]$ is the pressure, $x_j$ the mole fraction of the absorbing species, $T_j [\text{K}]$ the temperature, $S(T_j) \text{[cm}^{-2}\text{·atm}^{-1}]$ the line strength of the transition in the $j$-th pixel, respectively.

For a tomographic system with $M$ laser beams, (1) can be formulated as a linear equation

$$A k = b,$$  \hspace{1cm} (3)

where $A \in \mathbb{R}^{M \times N}$ is the sensing matrix, $k \in \mathbb{R}^N$ the vector of pixel-sized absorption density $k_j$ ($j=1, 2, \ldots, N$) to be solved in the inverse problem. $b \in \mathbb{R}^M$ is the measured integrated absorbance.

Due to the limited optical measurements in real application (i.e. $M < N$), the inverse problem described in (3) is inherently ill-posed.

As studied in [17] and [18], the sensing matrix $A$ can be illustrated through Singular Value Decomposition (SVD):

$$A = U S V^T,$$ \hspace{1cm} (4)

where $U \in \mathbb{R}^{M \times M}$ and $V \in \mathbb{R}^{N \times N}$ are orthonormal matrices, and $S \in \mathbb{R}^{M \times N}$ a diagonal matrix containing the singular values in descending order.

In principle, the inverse problem aims at finding a unique solution, $k^{LS} \in \mathbb{R}^N$ to minimize the least square error, that is, $k^{LS} = \arg \min \|b - U k^{LS} \|^2$. When $M > N$, this solution can be calculated by

$$k^{LS} = \sum_{j=1}^{N} \frac{u^T b}{\sigma_j} v_j,$$ \hspace{1cm} (5)

where $u_j$ and $v_j$ are the $j$-th column vectors of $U$ and $V$, respectively. $\sigma_j$ is the $j$-th singular value in the diagonal $S$. 

![Fig. 1. Geometric description of a line-of-sight TDLAS measurement in CST.](image-url)
In practice, the number of laser beams $M$ is limited by optical access to the combustors, resulting into $M < N$. In this case, $k^{LS}$ in Error! Reference source not found. should be separated into two parts, described in (6): the unique solution $k_{\text{unique}} \in \mathbb{R}^N$ that gives the minimized value of $\| Ak - b \|_2^2$ satisfying $Ak = b$, and the non-unique solution $k^{\text{null}} \in \mathbb{R}^{N-M}$ from solving $Ak = 0$.

$$k^{LS} = k_{\text{unique}} + k^{\text{null}}$$  \hspace{1cm} (6)

Considering the practical measurements $b$ is a superposition of noise-free data, $b^{\text{true}}$, and the noise, $b^{\text{noise}}$, $k_{\text{unique}}$ can be expressed by

$$k_{\text{unique}} = \sum_{j=1}^M u_j^T b^{\text{true}} v_j + \sum_{j=1}^M u_j^T b^{\text{noise}} v_j$$  \hspace{1cm} (7)

Since $A$ is rank-deficient, the lack of $N - M$ measurements causes nontrivial null-space in $A$, thus leading to $N - M$ undetermined values in $k^{\text{null}}$. Hence, (6) can be further expanded as:

$$k^{LS} = \sum_{j=1}^M \frac{u_j^T b^{\text{true}}}{\sigma_j} v_j + \sum_{j=1}^M \frac{u_j^T b^{\text{noise}}}{\sigma_j} v_j + \sum_{j=M+1}^N c_j v_j,$$  \hspace{1cm} (8)

where $c \in \mathbb{R}^{N-M}$, with the $j$-th element $c_j$, is a set of undetermined scalars to describe $k^{\text{null}}$.

Equation (8) mainly suffers from:

(a) Noise susceptible. As indicated by the second term in (8), $b^{\text{noise}}$ can be significantly magnified with relatively small singular values $\sigma_j$, worsening the quality of the reconstructed images.

(b) Solution underdetermined. In the case of $M < N$, the singular value does not exist for $j > M$. The third term in (8) is the indicator of undetermined solutions. The larger the $N - M$ is, the more severely underdetermined solution.

In the context of the Gauss-Newton algorithm, the image $k$ can be reconstructed iteratively by solving the regularized minimization problem [19]:

$$\arg\min \left\{ \| b - Ak \|_2^2 + \gamma \| Fk \|_2^2 \right\}, \text{ s.t. } k \geq 0,$$  \hspace{1cm} (9)

where $\| Fk \|_2^2$ is the first-order Tikhonov regularization term with a linear differential operator $F$, $\gamma$ the empirically determined regularization parameter. With $k$ in hand, the species concentration for each pixel, $x_i$, can be calculated from (2).

B. Size-adaptive hybrid meshing scheme

To alleviate the ill-posed problem mentioned above, we propose a 5-step size-adaptive hybrid meshing scheme to highlight the reconstruction of the target flow in the RoS. The RoS is divided into two parts: RoI that contains the target and the out-of-RoI region that reflects the background. It is worth mentioning that the dimension of RoI that covers the target flame can be determined in advance in both the lab-scale test [20] and industry application [15]. Although the flow field itself is dynamic, it can be located within the RoI. Therefore, the customized mesh sizes in and out of the RoI are adaptively determined, offline, and before the experiment, according to the spatial resolution. Fig. 2 shows the 5 steps for implementation of the proposed scheme plotted with small numbers of laser beams and pixels for a clear view of the meshes and beam arrangement. As exampled later in this subsection, the proposed scheme is also suitable for practical application with more densely arranged laser beams and finer discretized sensing regions.

Step 1: Choose RoI. RoI is chosen based on the target size, the location of target flow, and beam arrangement. The RoI, generally located within the beam-dense region, fully covers the target flow. In Fig. 2 (a), the RoS is defined as a square region with dimensions of $L \times L$, while the central RoI is defined by $L_{RoI}=L/2$ to cover the beam-dense region.

Step 2: Find the marginal dimension of uniform-size meshing in the RoI. This step aims at finding the minimum meshing size that enables all the pixels in the RoI to have beams passing through. The process is done by enumeration. Given the RoS is discretized into $N \times N$ pixels, the enumeration starts from $N = 1$. As $N$ increases with a step size of 1, the mesh size in the RoI gradually decreases. When the pixel number reaches $N_{m}$, all the pixels in the RoI have at least one beam passing through. $N_{m}$ is the target marginal dimension. Given $N = N_{m} + 1$, it is the first time that one or more pixels in the RoI has no beam passing through. This threshold can be found by summing up the chord lengths of all the $M$ beams $\sum_{j=1}^M A_{jj}$ in the $j$-th pixel.
Fig. 2 (b) shows the threshold for the given beam arrangement when the discretization of the RoI changes from \( N_a \times N_a \) to \((N_a+1) \times (N_a+1)\). The pixels are shaded as black and while for \( \sum_j A_{ji} = 0 \) and \( \sum_j A_{ji} \neq 0 \), respectively.

**Step 3:** Calculate average spatial resolution in RoI, noted as \( \delta_{ave} \). As demonstrated in Fig. 2 (c), the RoI is divided into multiple sectors for evaluating the local spatial resolution, noted as \( \delta_{sector} \). \( \delta_{ave} \) is calculated by averaging \( \delta_{sector} \) of all sectors. Previous studies on spatial resolution of laser tomography system [21, 22] proposed a quantified method for defining segmentation sectors in the sensing region based on theoretical estimation of spatial resolution. \( \delta_{sector} \) is evaluated by the “rise time”, i.e. interval between 10% and 90% maximum of the Edge Spread Function (ESF), extracted from the reconstruction of a sharp-edge inhomogeneity. For the ESF figure, the vertical axis is the normalized amplitude while the horizontal axis is the linear distance with the size of each mesh, \( l_m = L/N_a \), as unit.

**Step 4:** Calculate optimized mesh size in the RoI, \( l_{out} \). The averaged spatial resolution \( \delta_{ave} \) quantifies how fine the reconstruction can be spatially resolved in the RoI with the given beam arrangement [22]. \( l_m \) is derived by

\[
\delta_{ave} = \delta_{ave} \times l_m
\]

**Step 5:** Decide mesh size out of the RoI, \( l_{out} \). To minimize the rank deficiency of the CST inverse problem, \( l_{out} \) should be set by (a) making all the pixels in the out-of-RoI region have laser beams passing through and (b) reflecting the potentially existing background variations. For simplicity, \( l_{out} \) can be set as an integer multiply \( l_m \), e.g. \( l_{out} = 4 \times l_m \) shown in Fig. 2 (e).

The proposed 5-step size-adaptive hybrid meshing scheme enables three major advantages: First, the computational efficiency is maximized with a given beam arrangement by well balancing the sizes and numbers of meshes in and out of the RoI. Second, reconstruction in the RoI is highlighted with better spatial resolution, while the physical integrity of the out-of-RoI absorption is maintained. Third, the quality of the whole reconstructed image can be improved, verified later in Sections III and IV, as the hybrid-size meshing scheme significantly reduces the rank deficiency of the inverse problem. For some special cases where multiple flames are distanced, more segmentations may be needed in the sensing region. It is also noteworthy that the meshing scheme can be extended to three or more mesh sizes by following similar customization. For example, we should also firstly find the RoIs (step 1 in Fig. 2), and then find the marginal mesh sizes (three or more) of all the RoIs for the specific given beam arrangement (step 2). Similarly, the local spatial resolution \( \delta_{sector} \) and thus the averaged spatial resolution in each individual RoI \( \delta_{ave} \) should be quantified by following step 3 for each individual RoI. With \( \delta_{ave} \) in hand, the optimized mesh size for each RoI is determined in step 4. Finally, the mesh size for the background in step 5 should be set to be compatible with all the dimensions of the finer meshes in the RoIs. However, this requires more a priori information about the target flows, and pre-determined beam arrangement. It will also introduce a more complex computation of the sensing matrix \( A \).

To mathematically validate the above-mentioned benefits, an example is introduced using a 128-way CST sensor. As shown in Fig. 3, a parallel beam arrangement is used with 128 laser beams arranged in 4 equiangular projection angles, each angle with 32 equispaced parallel beams. This parallel beam arrangement is a cost-effective optical layout for CST, which has been widely adopted for the tomographic sensor design in practical applications [8, 22]. The angular spacing between each projection angle is 45°, while the beam spacing within a projection angle, noted as \( d \), is 0.4 cm. The distance between each pair of laser emitter and detector, noted as \( D \), is 36.8 cm, enclosing the octagonal RoS.

For this example CST sensor, the 5-step meshing scheme is detailed as follows: (a) To cover the target flow, the side length of the RoI is set as \( L_{RoI} = 18.4 \) cm. (b) The enumeration starts from \( N = 1 \). When \( N = 128 \), some pixels in the RoI have no laser beam passing through, giving \( N_m = 127 \) and thus \( l_m = 0.29 \) cm. (c) As illustrated in Fig. 2 (c), 38 rectangular sectors with dimensions of 2 cm × 3 cm are segmented in the RoI. A small rectangular phantom perpendicular to the length of the sector with the edge of phantom cutting the middle of the sector at half-width is used to provide a sharp edge for reconstruction [21]. The calculated \( \delta_{sector} \) values are shown in TABLE I. \( \delta_{ave} \) equaling to 3.13, is obtained by averaging the 38 \( \delta_{sector} \). (d) \( l_m \) is obtained by equation (10), giving \( l_m = 3.13 \times l_m = 0.91 \). For simplicity, the RoS is discretized into an integer number of pixels by taking \( l_m = 0.92 \) cm. (e) \( l_{out} \) is chosen to be \( 4 \times l_m = 3.68 \) cm for mitigating rank deficiency as well as reflecting the potentially existing background variations. As a result, dense pixels with dimensions of 0.92 cm × 0.92 cm are deployed in

**TABLE I**

| \( \delta_{sector} \) VALUES IN THE 38 SELECTED SECTORS. |
|---|---|---|---|---|---|---|---|---|---|---|
| No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| \( \delta_{sector} \) | 2.6 | 2.6 | 2.8 | 2.9 | 3 | 3 | 2.9 | 2.8 | 3.2 | 2.9 |
| No. | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| \( \delta_{sector} \) | 3.1 | 3.1 | 3.1 | 3.1 | 2.9 | 3.2 | 3 | 3.1 | 3.2 | 3.2 |
| No. | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| \( \delta_{sector} \) | 3 | 3.2 | 3.2 | 3.2 | 3.1 | 3.2 | 3 | 3.2 | 3.2 | 3.2 |
| No. | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| \( \delta_{sector} \) | 3.1 | 3.6 | 3.6 | 3.5 | 3.5 | 3.6 | 3.5 | 3.5 | 3.6 | 3.6 |

![Fig. 3. An example 128-way CST sensor with RoI in the beam-dense region.](image-url)
the RoI, as shown in Fig. 4 (a). Sparse pixels with dimensions of 3.68 cm × 3.68 cm are deployed out-of-RoI region. The RoS is discretized into 628 pixels, with 576 pixels in the RoI, noted as \( P_{\text{RoI}}^{H} \), and 52 pixels out of the RoI.

In comparison with the size-adaptive hybrid meshing, the RoS with uniform-size meshes are given in Fig. 4 (b). To achieve a similar number of the pixels with the hybrid-size meshing, the uniform-size meshing scheme discretizes the RoS in Fig. 4 (b) into 592 pixels with dimensions of 1.42 cm × 1.42 cm, from which 196 pixels are in the RoI, noted as \( P_{\text{RoI}}^{U} \). Compared with the hybrid-size meshing, the uniform-size meshing has 36 less pixels in the RoS. Given the same number of laser beams, the slightly larger number of pixels actually results in more undetermined solutions compared with the uniform-size meshing. However, as validated in Sections III and IV, the improvement achieved by the proposed size-adaptive meshing on the image quality significantly outweighs this defect.

Fundamentally, the hybrid-size meshing scheme modifies the structure of the sensing matrix \( \mathbf{A} \). Given \( M \) laser beams, the sensing matrix for the hybrid-size meshing scheme, noted as \( \mathbf{A}_{\text{hybrid}} \), is obtained by concatenating the sensing matrix in the RoI, \( \mathbf{A}^{in} \in \mathbb{R}^{M \times P_{\text{RoI}}^{H}} \), with that out of the RoI, \( \mathbf{A}^{out} \in \mathbb{R}^{M \times (N - P_{\text{RoI}}^{H})} \).

\[
\mathbf{A}_{\text{hybrid}} = \begin{bmatrix}
\mathbf{A}^{in} & \mathbf{A}^{out}
\end{bmatrix} = \begin{bmatrix}
\mathbf{a}^{in}_{1,1} & \cdots & \mathbf{a}^{in}_{1,P_{\text{RoI}}^{H}} & \mathbf{a}^{out}_{1,P_{\text{RoI}}^{H}+1} & \cdots & \mathbf{a}^{out}_{1,N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{a}^{in}_{M,1} & \cdots & \mathbf{a}^{in}_{M,P_{\text{RoI}}^{H}} & \mathbf{a}^{out}_{M,P_{\text{RoI}}^{H}+1} & \cdots & \mathbf{a}^{out}_{M,N}
\end{bmatrix}
\]

(11)

where \( \mathbf{a}^{in} \) and \( \mathbf{a}^{out} \) are the elements in \( \mathbf{A}^{in} \) and \( \mathbf{A}^{out} \), denoting the chord lengths of the laser paths in a pixel in and out of the RoI, respectively. Given \( P_{\text{RoI}}^{H} = 576 \) for the hybrid-size meshes shown in Fig. 4 (a), the numbers of columns in \( \mathbf{A}^{in} \) and \( \mathbf{A}^{out} \) are 576 and 52, respectively.

Similarly, the sensing matrix of the uniform-size meshing scheme, noted as \( \mathbf{A}_{\text{unifom}} \), can be described as:

\[
\mathbf{A}_{\text{uniform}} = \begin{bmatrix}
\mathbf{A}^{in} & \mathbf{A}^{out}
\end{bmatrix} = \begin{bmatrix}
\mathbf{a}^{in}_{1,1} & \cdots & \mathbf{a}^{in}_{1,P_{\text{RoI}}^{U}} & \mathbf{a}^{out}_{1,P_{\text{RoI}}^{U}+1} & \cdots & \mathbf{a}^{out}_{1,N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{a}^{in}_{M,1} & \cdots & \mathbf{a}^{in}_{M,P_{\text{RoI}}^{U}} & \mathbf{a}^{out}_{M,P_{\text{RoI}}^{U}+1} & \cdots & \mathbf{a}^{out}_{M,N}
\end{bmatrix}
\]

(12)

where \( \mathbf{A}^{in} \in \mathbb{R}^{M \times P_{\text{RoI}}^{U}} \) and \( \mathbf{A}^{out} \in \mathbb{R}^{M \times (N - P_{\text{RoI}}^{U})} \) are sensing matrices in and out of the RoI for the uniform-size meshing scheme, respectively. Given \( P_{\text{RoI}}^{U} = 196 \), the numbers of columns in \( \mathbf{A}^{in} \) and \( \mathbf{A}^{out} \) for the uniform-size meshes shown in Fig. 4 (b) are 196 and 396, respectively.

The performance of image reconstruction using the two meshing schemes can be quantified by plotting the singular values of the sensing matrices. For simplicity, the last nontrivial rows in both matrices obtained from the two meshing schemes are duplicated and extended into the dimensions of \( N \times N \), \( N \) equals to 628 and 592 for the hybrid-size and uniform-size meshes, respectively. As a result, both yield \( N \) singular values in the diagonal \( \mathbf{S} \). Fig. 5 shows the singular values \( \sigma_j (j = 1, 2, \ldots, N) \) in descending order obtained from each meshing scheme with the undetermined \( \sigma_j \) equaling to zero. Given \( j < 128 \), \( \sigma_j \) obtained using the hybrid-size meshes are not only larger than those obtained using the uniform-size meshes but also perform a slower decay to zero. As indicated by the second term of (8), the reduced number of small singular values imposes stronger suppression to the measurement noise, thus contributing to better image quality.

### III. Numerical Validation

Numerical simulation is firstly carried out to validate the proposed size-adaptive hybrid meshing scheme by reconstruction of two simulated phantoms. The reconstructed images are then quantitatively compared with those obtained using the uniform-size meshing scheme.

![Fig. 4. Discretized RoS with (a) hybrid-size meshes and (b) uniform-size meshes.](image)

![Fig. 5. Comparison between the singular values of the sensing matrices obtained using the hybrid-size meshing scheme with 628 pixels and the uniform-size meshing scheme with 592 pixels.](image)
A. Simulation setup

Water vapor (H\textsubscript{2}O) is one of the major combustion products and its distribution is interested in industrial combustion community for the evaluation of combustion efficiency. In this paper, the H\textsubscript{2}O transition centered at \( v = 7185.6 \text{ cm}^{-1} \) is selected to reconstruct the distributions of H\textsubscript{2}O concentration since it has appropriate line strength to give a good signal to noise ratio (SNR) for the TDLAS measurements [23].

Two phantoms of 2D distributions of the H\textsubscript{2}O concentration are generated with one and three inhomogeneities. Each inhomogeneity is simulated by a 2D Gaussian distribution, which can be expressed as

\[
X(x, y) = 0.01 + 0.1 \sum_{p=1}^{P} \exp \left[ - \frac{(x-x_p^c)^2 + (y-y_p^c)^2}{\sigma^2} \right],
\]

where \( x \) and \( y \) denote the horizontal and vertical coordinates of the RoS respectively. \((x_p^c, y_p^c)\) is the central coordinate of the \( p \)-th Gaussian distribution. \( P \) is the total number of Gaussian distributions. \( \sigma \) is the standard deviation. Two high-resolution phantoms with 0.07 cm \( \times \) 0.07 cm pixel dimension are generated for the CST forward problem to provide accurate modelling. The detailed parameters are shown in Table II.

B. Metrics for image quality quantification

In this work, three metrics are used to quantitatively evaluate the quality of the tomographic image [24].

Image Error (IE): IE is defined as the normalized root mean square error between the reconstructed and the true images:

\[
IE = \frac{1}{N} \sum_{j=1}^{N} \sqrt{\frac{(X_{rj}^{true} - X_{rj}^{rec})^2}{X_{rj}^{true}}},
\]

where \( X_{rj}^{true} \) and \( X_{rj}^{rec} \) refer to the reconstructed and true H\textsubscript{2}O concentration at \( j \)-th pixel, respectively. The concentration distribution is obtained by applying the meshes used in the reconstruction to the high-resolution phantom shown in Fig. 6. Dislocation (DL): DL characterizes the relative error of the centroid locations between the reconstructed inhomogeneity \((x_c, y_c)\) and that of true inhomogeneity \((x_{c, true}, y_{c, true})\). The centroid of the Gaussian-shaped inhomogeneity in the phantoms coincides with its center.

\[
DL = \sqrt{(x_c - x_{c, true})^2 + (y_c - y_{c, true})^2},
\]

Centroid Value Error (CVE): CVE calculates the relative difference of concentration values at the centroids of the reconstructed inhomogeneity \(X^{rec}(x_c, y_c)\) and that of true inhomogeneity \(X^{true}(x_{c, true}, y_{c, true})\).

\[
CVE = \frac{|X^{true}(x_c, y_c) - X^{true}(x_{c, true}, y_{c, true})|}{X^{true}(x_{c, true}, y_{c, true})}
\]

For phantoms with multiple inhomogeneities, CVE is calculated by the mean value of each Gaussian-shaped inhomogeneity.

C. Simulation results and discussion

In the simulation, the phantoms shown in Fig. 6 are reconstructed using both the proposed size-adaptive hybrid meshing scheme and the uniform-size meshing scheme. Since the regularization parameter \( \gamma \) in (9) plays an important role in image reconstruction [24, 25], the optimal \( \gamma \) is selected for each phantom based on quantification of IE in the RoI. Fig. 7 shows the dependence of IE on \( \gamma \) for the uniform-size and hybrid-size meshing schemes, respectively. The IE value is obtained by averaging the results from 50 repetitive reconstructions of each simulated phantom with the SNR of the line-of-sight TDLAS measurements set to 40 dB. When \( \gamma \) varies from \( 10^{-4} \) to \( 10^{4} \) with 32 uniform steps of logarithmic increment, the value of the IEs for each given \( \gamma \) is interpolated as the curve shown in Fig. 7. The optimal \( \gamma \) is selected where the minimum values of the mean IEs are obtained. The minimum mean IEs, i.e., IE = 0.24 and IE = 0.15, are obtained for the uniform-size and the hybrid-size meshing schemes given \( \gamma = 100 \) and \( \gamma = 50 \), respectively.

Fig. 8 shows the reconstructed images of the two phantoms in Fig. 6 with the two meshing schemes. Overall, the hybrid-size meshing scheme significantly outperforms the uniform-size meshing scheme in the following two aspects: (a) finer details of the inhomogeneities; (b) better retrieval accuracy of the pixel-wised concentration. Moreover, the performances of the two meshing schemes are quantified using simulated CST measurements contaminated with different levels of noise. Fig. 9 (a)-(c) show the average IEs, DLs and CVEs of reconstructed distributions of H\textsubscript{2}O concentration at different SNRs, respectively. The value of each IE, DL and CVE at a given SNR is the average of values obtained from reconstruction of the above-mentioned two phantoms. The

![Fig. 6. Simulated phantoms of 2D distributions of H\textsubscript{2}O concentration with (a) one homogeneity (b) three homogeneities, respectively.](image)

![Fig. 7. Dependence of IE on \( \gamma \) in the RoI for the uniform-size and hybrid-size meshing schemes.](image)
hybrid-size meshing scheme gives persistently lower values at all the metrics. For practical CST measurements with SNRs better than 35 dB [7], the hybrid-size meshing lowers IE by 35%. In addition, hybrid-size meshing lowers the DL by about 38% at all given SNRs, compared with those obtained by the uniform-size meshing. As shown in Fig. 9 (c), the hybrid-size meshing gives CVE values one order of magnitude lower than those obtained using the uniform-size meshing, indicating better accuracy of the retrieved peak values [24].

IV. EXPERIMENTAL VALIDATION

The effectiveness of the proposed size-adaptive hybrid meshing scheme is further examined by reconstructions of practical H₂O evaporation processes. The practical distributions of H₂O concentration in diffusion flows are acquired by Large Eddy Simulation (LES) via Fire Dynamic Simulator (FDS) [26, 27]. The experimental domain is a 36.8 × 36.8 × 10 cm³ rectangular space discretized into 280 × 280 × 20 pixels. The domain is filled with air with its top and four side boundaries opened. The temperature and H₂O mole fraction in the background are considered uniform with 294.15 K and 0.01, respectively. To generate the inhomogeneous H₂O distributions, water vapor is jet from a circular inlet at the bottom of the domain with a constant velocity. The cross section for CST is set at 1 cm above the inlet. In this work, two scenarios are considered, one with the inlet located at the center, the other with the inlet located to the bottom right of the center. Table III details the inlet radius, r [cm], central location of the inlet, (x, y) [cm], and the jet velocity, v [m/s] for each scenario. To obtain high-accuracy path integrated absorbances b in (3), two sets of high-resolution phantoms are generated with 10,136 pixels in the RoS, each with 0.13 cm × 0.13 cm. Measurements are taken with 35 dB SNR to consider practical noise [7]. For both scenarios, the videos in Media 1 and Media 2 start by jetting the water vapor from the inlet and record a total of consecutive 50 frames sampled at an interval of 0.2 s. Two instantaneous distributions of H₂O concentration, the 35th frame of the first scenario and 15th of the second scenario are shown in Fig. 10 (a) and (d), as representative demonstrations.

As detailed in Section III. B, the optimal regularization parameters γ in (9) are set as 100 and 50 for the uniform-size and hybrid-size meshing schemes, respectively. As shown in Figs. 10 (b) and (e), reconstructed H₂O distributions using the hybrid-size meshing scheme reveals more details of the H₂O vapor jet diffusion in the RoI and have fewer artifacts in the background, compared with those obtained using the uniform-size meshing scheme. For scenario 1, the peak value in Fig. 9 (b), 0.07, is closer to the true value of 0.072 in Fig. 10 (a), indicating more convincing retrieval of the H₂O concentration. The finer resolved pixels in the RoI also enables more clearer boundary of the inhomogeneity, compared with the over diffused boundary in Fig. 10 (c). For scenario 2, the hybrid-size meshing scheme is superior for revealing the profile of the H₂O inhomogeneity, with the diffusion towards upper right of the sensing region clearly observed in Fig. 10 (e). In contrast, the uniform-size meshing can only give a blur and expanded shape of the target inhomogeneity in Fig. 10 (f) with the absolute H₂O concentration values severely deviated from the truth. Both cases indicate that the reconstructed images using the hybrid-size meshing can characterize the gradient of H₂O concentration around the vapor injector, while those reconstructed using the uniform-size meshing suffer from a much lower peak value and a severely blurred boundary of the inhomogeneity. Therefore, the experimental results show the adapted hybrid mesh size for CST is better at detailing the profiles of the target flows with improved quality of reconstructed images.

As shown in Media 1 and Media 2, the consecutively reconstructed images using the proposed hybrid-size meshing scheme can reveal more structural information about the cross-sectional variation of H₂O concentration. For the whole

![Fig.8. Reconstructions of the phantoms in Fig. 6 using (a, b) the proposed size-adaptive hybrid meshing scheme and (c, d) uniform-size meshing scheme.](image-url)

![Fig.9. Comparison of image error in (a) RoI (b) Dislocation (c) Centroid value error with uniform-size and hybrid-size meshing schemes at different measurement SNRs.](image-url)
H$_2$O evaporation process, the absolute concentration of the inhomogeneity retrieved by the hybrid-size meshing scheme is closer to the truth. Table IV shows the quantitative performance evaluation, e.g., IE, DL, CVE, of the two scenarios averaged for 50 frames using the proposed hybrid-size meshing and uniform-size meshing schemes, respectively. For the both scenarios, the hybrid-size meshing gives about 8% lower IE, and halves DL and CVE than the uniform-size meshing, which indicates better accuracies for pixel-wised reconstruction, target location and peak value.

V. CONCLUSION

A size-adaptive hybrid meshing scheme was proposed to finer resolve the reconstruction of the target flow fields with better image quality and effective computational cost. Driven by the required spatial resolution, 5 steps were proposed in this work to quantitatively customize the mesh size, and thus the number of meshes, in and out of the ROI. This customization alleviates the rank deficiency in the CST inverse problem by reducing undetermined solution and smoothing the distribution of singular values, resulting in better accuracy of the tomographic reconstruction.

The 5-step size-adaptive hybrid meshing scheme was numerically demonstrated by applying to a 128-beam CST sensor. It was firstly validated by reconstructing phantoms of 2D Gaussian distributions of H$_2$O concentration using both the proposed hybrid-size and the traditionally uniform-size meshing schemes. The numerical results show the superiority of the size-adaptive hybrid meshing scheme with 35% lower image error and 38% less significant dislocation at typical 35 dB SNR, as well as the more accurate peak values, compared with the uniform-size meshing. Furthermore, experimental validation was carried out by consecutive reconstruction of practical transpiration-introduced water vapor, which is generated by FDS. The reconstruction using the size-adaptive hybrid meshing scheme significantly outperforms the uniform-size meshing, with finer details, clearer boundary and more accurate concentration of the target flow fields.

ACKNOWLEDGEMENT

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TABLE III

<p>| Simulation Parameters of the Two Scenarios. |
|-----------------------|---------------------|---------------------|</p>
<table>
<thead>
<tr>
<th>$r$ [cm]</th>
<th>$(x,y)$ [cm]</th>
<th>$v$ [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>5</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>4</td>
<td>(-3, -3)</td>
</tr>
</tbody>
</table>

TABLE IV

Quantitative comparison between the hybrid-size and uniform-size meshing schemes for two scenarios.

| Meshing | IE | DL [cm] | CVE |
|-----------------------|---------------------|---------------------|
| Scenario 1 | Uniform | 0.32 | 1.13 | 0.27 |
| Hybrid | 0.28 | 0.54 | 0.12 |
| Scenario 2 | Uniform | 0.50 | 1.23 | 0.45 |
| Hybrid | 0.46 | 0.68 | 0.20 |

Fig. 10. Reconstruction of (a, d) two experimental H$_2$O phantoms simulated via FDS using the (b, e) hybrid-size and (c, f) uniform-size meshing schemes.

REFERENCES


